The Electrostatic Field and Electrostatic Potential Relationship
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This is the fourth in a series of articles dealing with coaxial cables operating in the frequency span from direct current through the microwave region. The first article, which dealt only with currents, inductance, and magnetic fields in coaxial cables, was intended to be a stand-alone article directed toward a readership that was assumed to be familiar with the mathematics and physics of scalar and vector fields. Subsequently, the author was urged to cover both the electric as well as magnetic properties of the cable from both a circuit as well as field point of view. In order to accomplish this in a meaningful way, now for a wider readership, the second article was entitled “Mathematics Primer for Vector Fields”. This article treated the general mathematics of vectors as well as vector and scalar fields and concluded with the introduction of the gradient theorem of a scalar field. The third article in the series was entitled “Gauss’s Law and the Electrostatic Field” in which the divergence theorem of vector analysis plays a paramount role. This article concluded with the calculation of the electrostatic potential difference as well as the capacitance between the center and outer conductors of a section of charged coaxial cable. The first order of business in the present article is to justify this calculation. In doing so it is necessary to introduce another theorem of vector analysis known variously as the curl, circulation, or Stoke’s theorem.

As a warm-up to the curl theorem we will first work through a sample calculation of the line integral around a closed path of a given vector field. The field in question is a fluid velocity field given by $\mathbf{v} = 2yz\,\hat{j}\ \mathrm{m^{-1}sec^{-1}}$. This describes a fluid velocity whose direction is that of the y-axis, whose value depends on the product of the y and z coordinates expressed in meters, and whose ultimate dimensions are meters per second. These ultimate dimensions are obtained when y and z expressed in meters are substituted into the expression for **v.** We want to calculate the line integral or circulation of this field around the perimeter of a square that is one meter on a side as depicted in Fig. 1.

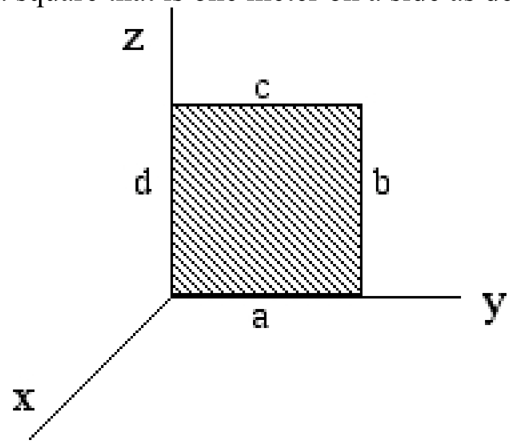

Figure 1. The closed path of circulation consists of sides a, b, c, and d each of one meter.

The portion of the path designated as a in the figure lies along the y-axis and extends from the origin to y equal to one meter. The z coordinate is everywhere zero along this segment so **v** is also zero along a. Consequently, $\int \mathbf{v} \bullet d\boldsymbol{l} = \int_0^1 \mathbf{v} \bullet (dy\,\hat{j}) = 0$.

Proceeding around the perimeter in a counterclockwise fashion we next encounter the side b of the figure. Here we are proceeding in the direction of increasing z so $d\boldsymbol{l} = dz\,\hat{k}$. The fluid velocity along b has a value for all z>0, but it exists only in the $\hat{j}$ direction so **v**•d***l*** is everywhere zero along side b. Next we encounter side c that runs from the point y=1, z=1 to the point y=0, z=1. Along this portion, $d\boldsymbol{l} = dy\,\hat{j}$ and $\mathbf{v} = 2y\,\hat{j}$ so the contribution to the line integral becomes $\int_1^0 (2y\,\hat{j}) \bullet (dy\,\hat{j}) = \int_1^0 2y\,dy = -1$. The dimensions of this result will be $m^2\,sec^{-1}$ as we have in effect multiplied a velocity by a length. Finally, we have arrived at side d where the value of the y coordinate is uniformly zero that in turn forces the fluid velocity to be zero all along d. Again there is no contribution to the line integral anywhere along d. Therefore the line integral around the closed path abcd becomes just that which occurred along c or $\oint \mathbf{v} \bullet d\boldsymbol{l} = -1\ m^2sec^{-1}$. Since we circulated around the closed path in the counterclockwise direction we will describe the plane area of the square by a vector perpendicular to the plane containing the square and pointing in the direction of the x-axis. This is so because a right-handed screw rotated counterclockwise will advance along the x-axis in Fig. 1.

The next aspect of the curl theorem concerns the calculation of the flux of the curl of the vector field through a surface area that is bounded by a closed path. We defined both a path and a surface area in Fig. 1. Now we must calculate the curl of our given vector field. The curl of **v** is given by the determinant

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2yz & 0 \end{vmatrix} = -2y\,\hat{i}.$$

Finally, we calculate the flux of the curl of **v** through the area of the square depicted in Fig. 1. This flux is the result of the following surface integral.

$$\int (\nabla \times \mathbf{v}) \bullet d\mathbf{S} = \int_0^1 \int_0^1 (-2y\,\hat{i}) \bullet (dz\,dy\,\hat{i}) = -1$$

The curl of **v** has the dimensions of sec^{-1} so the flux of the curl of **v** dimensionally is m^2sec^{-1}. Our final result, then, may be stated as

$$\oint \mathbf{v} \bullet d\boldsymbol{l} = \int (\nabla \times \mathbf{v}) \bullet d\mathbf{S}.$$

The curl theorem is equally valid for all vector fields and when stated for the electrostatic field in particular appears in the form

$$\oint \mathbf{E} \bullet d\boldsymbol{l} = \int (\nabla \times \mathbf{E}) \bullet d\mathbf{S}.$$

In words the theorem states that the line integral of the electrostatic field around <u>any</u> closed path is equal to the curl of the electrostatic field integrated over <u>any</u> surface having the closed path as a boundary. From the third article in this series we learned that the electrostatic field of a single charged particle is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.$$

In writing the above equation we have taken the origin at the location of the charged particle and expressed the field at a distance r from the charge employing spherical polar coordinates with $\hat{r}$ being the radial unit vector. You will recall that this field is diverging radially so that it is independent of the polar angle θ and the azimuthal angle ϕ. The physics of any problem is independent of the choice of coordinates although some coordinate systems may be more convenient than others. For example, in Cartesian coordinates the same equation would appear as

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}}.$$

It should now be obvious why spherical polar coordinates are more suitable in this instance! Now we will apply the left hand side of the curl theorem by calculating the line integral of **E**•d***l*** around an oval closed path as depicted in Fig. 2. We will form a closed path by integrating from a to b then from b to c and finally from c back to a. Note that **E**•d***l*** is given by $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \bullet (dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr.$

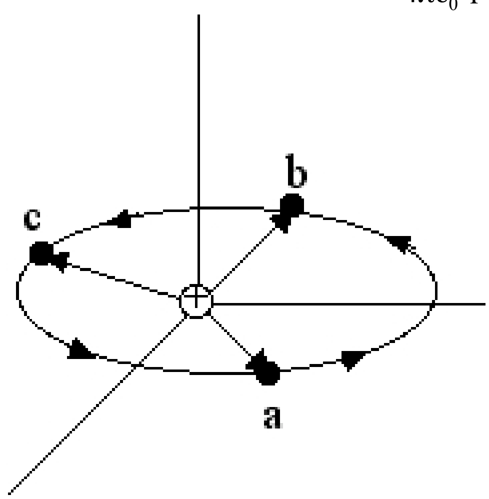

Figure 2. An arbitrary closed path of integration in the presence of a point charge located at the origin of coordinates. The various radial lines from the origin are r_a, r_b, and r_c to the points a, b, and c, respectively.

$$\oint \mathbf{E} \bullet d\boldsymbol{l} = \frac{1}{4\pi\varepsilon_0}q\left[\int_a^b \frac{dr}{r^2} + \int_b^c \frac{dr}{r^2} + \int_c^a \frac{dr}{r^2}\right] = \frac{-q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a} + \frac{1}{r_c} - \frac{1}{r_b} + \frac{1}{r_a} - \frac{1}{r_c}\right] = 0.$$

Any path for which the starting and ending points were the same distance from the origin would yield an identical result. The surface over which the integration on the right hand side of the statement of the curl theorem is carried out can have any shape as long as it has the closed path as a boundary. The only way that this surface integral can vanish for all possible surfaces having the given boundary and thus satisfy the curl theorem is if the curl of the electrostatic field of a charged particle is identically zero everywhere. We have now learned two additional things about the electrostatic field of a charged particle. The line integral of this field between two points is independent of the chosen path between the points and the curl of this field is zero.

Even though the proof given above was for a single charged particle located at the origin of our choice of coordinates, the properties of the field are independent of our choice of origin and indeed of the particular coordinate system that we choose to employ. Furthermore, the strength of the field of a single charged particle is linear in terms of the amount of charge, that is, q appears to the first power. This linear behavior means that if one has a collection of charged particles we can apply the principle of superposition in arriving at the expression for the total electrostatic field of the ensemble of particles so the total field is the vector sum of the individual fields or

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots$$

and

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \cdots) = \nabla \times \mathbf{E}_1 + \nabla \times \mathbf{E}_2 + \cdots = 0 .$$

For any electrostatic distribution of charge whether it be a single charged particle, a collection of individual charge particles, or charged particles so densely packed as to form a continuum on a surface or throughout a volume; the following two equations will apply.

$$\oint \mathbf{E} \bullet d\boldsymbol{l} = 0$$
$$\nabla \times \mathbf{E} = 0 .$$

The fact that the curl of the electrostatic field is always zero is a particularly significant result in that it means that the electrostatic field can always be expressed as the gradient of a scalar field. Why is this so? The answer is very simple because the curl of the gradient of a scalar field is always identically equal to zero! We will give a general example of this. For reasons that will be apparent presently, we choose to set $\mathbf{E} = -\nabla V$ where V is a scalar function of Cartesian coordinates, i. e., V=V(x,y,z). First we calculate the negative of the gradient of V to obtain $\mathbf{E} = -(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k})$. Next we calculate the curl of $\mathbf{E}$ to obtain $-\left[(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y})\hat{i} + (\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z})\hat{j} + (\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x})\hat{k}\right] \equiv 0$. The reason that the term in the brackets is identically equal to zero is that the order of taking repeated partial differentiations is immaterial and as result each term in a parenthesis within the brackets is zero. What is the physical nature of this scalar field V from which we may extract the electrostatic field **E** simply by taking the negative of the gradient of V? The answer is the subject of our next example.

Only for reasons of mathematical simplicity we again start with a single positively charged particle of charge amount q that is fixed at the origin of a spherical polar coordinate system. This charged particle will constitute the source of our electrostatic field $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$. Additionally we have a second, small positively charged particle of charge amount q_t that is movable and is initially so far removed from the position of the source of the field that for all practical purposes is infinitely far away. This second charged particle is termed the test particle and is initially at rest. This implies two things. The initial resting state means that the test particle's kinetic energy is initially zero and the remote location means that there is no interaction with the fixed charge at the origin and hence there is initially no potential energy as well. What we want to do is to slowly move the test particle along a radial path towards the source of the electrostatic field and calculate the work that we must perform in accomplishing this task. Once we get the test particle moving ever so slowly, the force that we must exert at each point along the way must be equal in magnitude while oppositely directed to the force exerted on the test particle by the electrostatic field. At each point along the radial path the force that we exert is then $-\mathbf{E}q_t$. This force is in the negative radial direction and since we are decreasing the radial distance between q and q_t the displacement that we produce is also in the negative radial direction so **F**•d***l*** is always positive. Consider that the initial location of q_t is at an infinite distance and that the final resting location of q_t is at a coordinate point called the point a, that is separated from the fixed charge by the radial distance r_a. So, the work we have performed and the potential energy acquired by the system in the process is given by

$$\int_i^f \mathbf{F} \bullet d\boldsymbol{l} = \int_\infty^{r_a} -\frac{1}{4\pi\varepsilon_0}\frac{qq_t}{r^2}dr = \frac{1}{4\pi\varepsilon_0}qq_t\left[\frac{1}{r_a}-\frac{1}{\infty}\right] = \frac{1}{4\pi\varepsilon_0}\frac{qq_t}{r_a}.$$

Now if we divide this result by the size of the charge that we physically moved in the process we obtain what is called the electrostatic potential at the coordinate point as a result of the fixed charge q at the origin of coordinates.

$$V_a = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_a}$$

What are the dimensions of this quantity? The line integral above has the dimensions of work or Joules while the electrostatic potential, then, must have the dimensions of work divided by charge or Joules per Coulomb. This is called a Volt. There is nothing magic about the point a as it could be any point at a radial distance r from the origin. Therefore, the electrostatic potential function for this simplest of cases expressed in spherical polar coordinates is

$$V = V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

The same expression in Cartesian coordinates has the form

$$V = V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(x^2+y^2+z^2\right)^{1/2}}.$$

Remember that the property of the scalar electrostatic potential is such that if you know the potential for a given charge distribution then you may extract the electrostatic field for that charge distribution from $\mathbf{E} = -\nabla V$. Let's establish that this is so by using the two expressions for the potential of a point charge given above. In spherical polar coordinates when the potential depends only on the radial coordinate the gradient operator is $\nabla = \hat{r}\frac{\partial}{\partial r}$ so that $\nabla V = \hat{r}\frac{\partial V}{\partial r} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$. **E** of course is the negative of this so $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$.

Similarly, but requiring more effort, in Cartesian coordinates the gradient operator is $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ and $\mathbf{E} = -\nabla V = \frac{1}{4\pi\varepsilon_0}\frac{q(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{3/2}}$. Finally, lets consider two separated points called i for initial and f for final in the electrostatic field of our positively charged particle. Let the point i be closer to the particle while point f is further away. The potential at the initial point is V_i while that at the final point is V_f and the potential difference between the points is $V_{if} = V_i - V_f > 0$. This means that a positively charged test particle placed at the point i will have a greater potential energy than when placed at point f. Now let's calculate the work done per unit of positive test charge by the electrostatic field as it moves the test charge from the initial point to the final point.

$$\int_i^f \mathbf{E} \bullet d\boldsymbol{l} = \int_i^f -\nabla V \bullet d\boldsymbol{l}$$

We learned from the gradient theorem presented in the second article in this series that

$$\int_i^f \nabla V \bullet d\boldsymbol{l} = \int_i^f dV = V_f - V_i$$

In comparing the two equations given immediately above we are led to the conclusion that

$$\int_i^f \mathbf{E} \bullet d\boldsymbol{l} = -(V_f - V_i) = V_i - V_f = V_{if}.$$

The conclusion is that the electrostatic field acts so as to always move a positive charge from a position of high potential energy per unit charge towards a position of lower potential energy per unit charge. The equation immediately above is the one employed in the calculation that concluded the previous article in this series.

You very well might reasonably ask why we are always talking about positive charge when we know that in metallic conductors only negative charges in the form of electrons are the mobile carriers of charge. The answer is quite simple. The early scientists such as Gilbert, Franklin, Oersted, Faraday, Ampere, Maxwell, and a host of others performed their investigations long before the properties of the fundamental charged particles were ever discovered much less the details of the conduction process on a microscopic scale. These scientists had to establish a common language and set of definitions in order to communicate their results among themselves as well as others in order to make progress in studying the phenomena. The choice of action on or by positive charge in deciding on the direction of an electrostatic field was an arbitrary agreed upon choice. In fact, electrical current is a scalar quantity and has no direction whereas the current density, that is charge per unit area per unit time, is a vector in the direction of the electrostatic field at the point in question. We will have more on this later on. Even so, in gaseous conductors one has positive ions as well as free electrons in motion in opposite directions under the influence of the local electrostatic field. In electrolytes or liquid conductors one has both positive and negative ions moving in opposite directions under the influence of the local electrostatic field. The conclusion is that positive charge moves in the direction of the electrostatic field whereas negative charge will move in the opposite direction. In both instances regardless of the sign of the mobile charges the motion under the influence of the electrostatic field will be such as to reduce the potential energy of the respective charges. The electrostatic potential itself is always a function of the spatial coordinates and may be either positive or negative at a given point dependent upon the nature of the charge distribution that is the source of the field. The potential energy possessed by a given charge at some space point, however, depends on the product of the potential at that point with the respective charge or Vq. Here is an example that may make you feel a little more comfortable. Suppose you have two neighboring space points A and B with the electrostatic potential at A being 5 volts while at B the electrostatic potential is 2 volts. Clearly, V_{AB} is 5-2=3 volts and the electrostatic field is directed from A to B. A proton whose charge is $1.6(10^{-19})$ Coulomb when placed at A will have a potential energy of $8(10^{-19})$ Joule and after having been moved by the electrostatic field to the point B will have a potential energy of $3.2(10^{-19})$ Joule so that its potential energy will decrease by $4.8(10^{-19})$ Joule as a result of the move under the influence of the electrostatic field. Instead let's now consider an electron. An electron has a charge of $-1.6(10^{-19})$ Coulomb and when placed at B has a potential energy of $-3.2(10^{-19})$ Joule. As the electron's charge is negative the electrostatic field exerts a force on the electron and moves it in the opposite direction of the field to the point A where the electron has a potential energy of $-8(10^{-19})$ Joule so that its potential energy will decrease by $4.8(10^{-19})$ Joule as a result of the move under the influence of the electrostatic field. In both instances the oppositely charged particles are moved by the field so as to decrease their respective potential energies even though their individual motions are in opposite directions with the proton moving from high electrostatic potential to low electrostatic potential while the electron moves from low electrostatic potential to high electrostatic potential.

Finally, we must point out that the electrostatic field acting alone can not maintain the motion of charges whatever their sign in a closed conducting path because the consequence that $\nabla \times \mathbf{E} = 0$ is that $\oint \mathbf{E} \bullet d\boldsymbol{l} = 0$. A closed conducting path begins and ends at the same point for which the potential difference is zero. In order to maintain the motion of charges in a closed path of whatever nature some agency must be present that can convert some other form of energy into electrical energy. This will be the subject of a future article.

In the next article in the series we will study the properties of dielectrics and discuss what happens when such materials fill the space between the coaxial conductors of our cable sample.