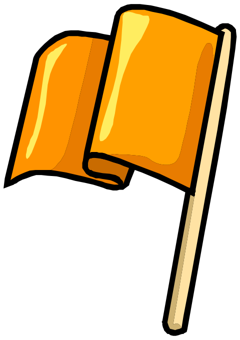

BY DR. EUGENE PATRONIS

What is Waving and Why

The Anatomy of the Wave Equation - Part 6

In part 5 of this series we employed the method of images to determine the complex mechanical impedance present at the origin of a plane wave tube of arbitrary length when the tube is terminated by a rigid barrier. In this article we will explore the general technique that is applicable to plane wave tubes having arbitrary terminations including that of a rigid barrier, an open-ended tube, and any other given mechanical impedance. Finally, we will consider the interaction between a small loudspeaker employed to excite the tube and an improperly terminated plane wave tube.

As a reminder, the mechanical impedance is defined to be the ratio of the complex mechanical force applied to an object divided by the resulting complex mechanical velocity of the object

$$Z_m = \frac{F}{u}$$

Additionally, at any point in a sound field the ratio of the complex acoustic pressure to the resulting particle velocity is called the specific acoustic impedance at the point in question

$$Z_s = \frac{p(z,t)}{u(z,t)}$$

Now consider a plane wave tube that is fitted with a piston at z = 0 and mechanical impedance of Z_L at the spatial point z = L. Let the piston displacement at any time be described by the phasor

$$\xi = \xi_m e^{j\omega t}$$

where ξ_m is the amplitude of the piston displacement and ω is the angular frequency of piston oscillation. Remember that the actual piston motion is given only by the real part of this phasor namely $\xi = \xi_m \cos(\omega t)$. In the general case Z_L does not properly terminate the tube so we must allow for both a primary wave and a reflected wave. In which case the phasor description of the two waves becomes

$$\xi(z,t) = Ae^{j(\omega t - kz)} + Be^{j(\omega t + kz)}$$

At the origin where z = 0, the boundary condition is satisfied by having this last expression match the given piston motion from which it is learned that $\xi_m = A + B$ Another independent equation is required in order to determine A and B uniquely. This equation is obtained by recognizing that at z = L the ratio of the acoustic force to the particle velocity at that point must be equal to the mechanical impedance at that point. It is necessary then to write the general expressions for the acoustic pressure and the particle velocity that are valid anywhere in the tube and particularly at z = L.

$$u(z,t) = \frac{\partial \xi(z,t)}{\partial t} = j\omega\xi(z,t) = j\omega\left[Ae^{j(\omega t - kz)} + Be^{j(\omega t + kz)}\right]$$

$$p(z,t) = \rho_0 c^2 s(z,t) = -\rho_0 c^2 \frac{\partial \xi(z,t)}{\partial z} = \rho_0 c^2 jk\left[Ae^{j(\omega t - kz)} - Be^{j(\omega t + kz)}\right]$$

where s(z, t) is the condensation and ρ_0 is the undisturbed air density. Now the force at z = L is the acoustic pressure at that point multiplied by the cross-sectional area, S, of the tube. Dividing the force by the particle velocity at z = L leads to the second equation involving A and B.

$$Z_L = \rho_0 c S \frac{\left(Ae^{-jkL} - Be^{jkL}\right)}{\left(Ae^{-jkL} + Be^{jkL}\right)}$$

The two independent equations for A and B are now solved simultaneously to obtain

$$A = \frac{\xi_m}{2}\,\frac{\left(1 + \frac{Z_L}{\rho_0 cS}\right)\left[1 + j\tan(kL)\right]}{1 + j\frac{Z_L}{\rho_0 cS}\tan(kL)}$$

and

$$B = \frac{\xi_m}{2}\,\frac{\left(1 - \frac{Z_L}{\rho_0 cS}\right)\left[1 - j\tan(kL)\right]}{1 + j\frac{Z_L}{\rho_0 cS}\tan(kL)}$$

Knowing the values for A and B it is now possible to evaluate ξ(z, t), p(z, t), and u(z, t) anywhere in the tube. In particular, at the input of the tube where z is zero we can determine the mechanical load or impedance that the tube presents to the motion of the piston. This term is called Z_0 and is calculated to be

$$Z_0 = \frac{Sp(0,\ t)}{u(0,\ t)} = \frac{Z_L + j\rho_0 cS\ \tan(kL)}{1 + j\dfrac{Z_L}{\rho_0 cS}\tan(kL)}$$

This last result is quite general and applies not only to tubes but other shapes as well as long as the operating wavelength is large compared with the largest dimension associated with the structure's cross-section. Our previous result for a tube terminated with a rigid barrier that was calculated in part 5 of this series of articles can be readily obtained from the general expression for Z_0 by dividing both numerator and denominator by Z_L and then allowing Z_L to approach infinity. Another observation with regard to this general case that is worthy of note is the behavior that occurs when the tube is driven at a frequency or frequencies such that the tube length is an integral number of half wavelengths. When this is true, the tangent terms in Z_0 are exactly zero and Z_0 becomes identically equal to Z_L and the tube's mechanical impedance opposing the piston's motion is the same as the mechanical impedance that terminates the tube. The half wavelength tube then acts as an ideal transformer having a turns ratio of 1:1.

Rather than being terminated in a rigid barrier or cap, suppose that the tube just ends abruptly at Z = L while being surrounded by a very large, ideally infinite, plane baffle. What is the terminating mechanical impedance in this instance? The answer is not zero because the air particles at the end of the tube must push against the outside air contained within a 2π solid angle when they suffer displacement by the forward traveling wave contained within the tube. In fact, the air particles at the end of the tube experience exactly the same impedance as that experienced by the front face of a piston that is radiating into a half space or 2π solid angle. Alternatively, the truncated end of the tube might just end in open space in which case the radiation is almost unconfined or experiences nearly a 4π solid angle. In the latter case the acoustic pressure is approximately one half of that of the former case. Since the force is directly proportional to the pressure, the impedance experienced by the truncated tube less the baffle is also approximately one-half that of the infinite baffle case. In either case, the terminating impedance is calculated through the employment of what is termed the piston impedance function. The piston impedance function has real and imaginary parts that are written as $R(2ka) + jX(2ka)$ where $k = 2\pi / \lambda$ and a is the piston radius or, in this case, the inner radius of the tube. The real part of the piston impedance function can be expressed in terms of the first order Bessel function of the first kind that we encountered in part 5 while the imaginary part can be expressed in terms of the first order Struve function.

$$R(2ka) = 1 - \frac{J_1(2ka)}{ka}$$

$$X(2ka) = \frac{H_1(2ka)}{ka}$$

These functions are graphed in Fig. 1.

The terminating impedance of a truncated tube with a baffle expressed in terms of the piston impedance function is $Z_L = \rho_0 cS[R(2ka) + jX(2ka)]$ while that of the truncated tube without a baffle is approximately one half this amount.

It is very important to note that all of the foregoing takes no account of energy losses occurring within the air or at the interior surfaces of the tube. Air has both a viscous shear modulus that is a loss factor at the tube surfaces and a viscous bulk modulus that is a loss factor throughout the enclosed volume. Heat generation and conduction in the body of the gas and at the tube walls are even further considerations with regard to energy loss. A pursuit of these topics would carry us much further into the physics of fluids than we are prepared to go here. Even though the losses are

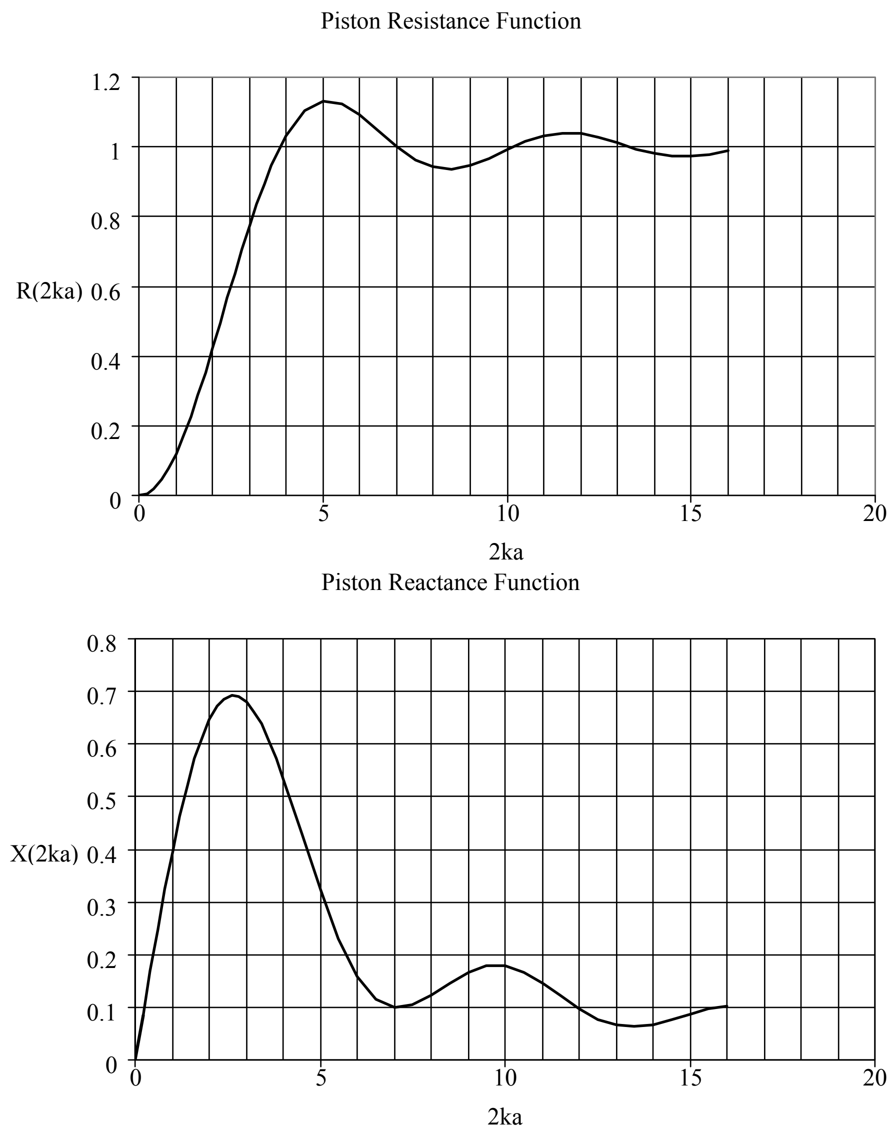

Figure 1. Real and Imaginary parts of the piston impedance function.

small for short tubes of reasonable diameter, their inclusion significantly complicates the mathematics of the description of the process. The losses could be accounted for in our equations by allowing the propagation constant k to be complex with the form $\beta - j\alpha$. Replacing k in our equations by this complex form forces the particle displacement in our description to become

$$\xi = Ae^{-\alpha z}e^{j(\omega t - \beta z)} + Be^{+\alpha z}e^{j(\omega t + \beta z)}$$

In addition to having to redo the analysis employing this starting point, the problem is further complicated by the fact that both α and β are frequency dependent in a complicated fashion. The frequency dependence of β is particularly troublesome because the phase velocity being ω / β will no longer be independent of frequency. The problem can be handled exactly but the mathematics is more tedious. We will consider our results to be a first as well as useful approximation to the more exact ones.

Now we will use our approximate results to calculate the lowest resonant frequencies of a two-inch diameter loudspeaker whose front face is attached to a short tube of two inches inner diameter with the far end of the tube being terminated in a rigid cap. The back of the loudspeaker is enclosed by a small box. The air trapped in the box and the loudspeaker's suspension together act as a spring with a total stiffness of K. The suspension also furnishes a mechanical resistance R_m. The moving mass of the loudspeaker cone is M and the loudspeaker itself has a total mechanical impedance Z_{ls} with $Z_{ls} = R_m + j(\omega M - K/\omega)$. As we learned in part 5 the closed tube loads the front face of the loudspeaker with a mechanical impedance that is $-j\rho_0 cS\cot(kL)$. The total mechanical impedance presented to the agency that drives the loudspeaker is then the sum of these two impedances with $Z_m = R_m + j(\omega M - K/\omega - \rho_0 cS\cot(kL))$. The driven loudspeaker will be at resonance for those frequencies where the total reactance in the mechanical impedance expression becomes zero or where $\omega M - K/\omega = \rho_0 cS\cot(kL)$. In this last equality, we replace k by ω/c and then replace ω by $2\pi f$ so that both sides can be plotted versus f. The intersection points of the two resulting curves identify the resonant frequencies. This was done by employing the parameters typical of a two-inch loudspeaker mounted on a two-inch tube of one-meter length. The results are presented in Fig. 2. For comparison purposes, the same calculation was performed taking account of air losses in the tube. These results are presented in Fig. 3.

The mechanical impedance at the input of a capped tube when the air losses in the tube are small is

$$Z_0 = \rho_0 c S \frac{\alpha L - j\cos(kL)\sin(kL)}{\sin^2(kL) + (\alpha L)^2\cos^2(kL)}$$

The real part of this expression is a mechanical resistance and along with the mechanical resistance of the loudspeaker broadens the shape of a resonance but does not effect its location. Setting α equal to zero reduces the impedance expression to that which was derived without considering air loss. There are resonances beyond the frequency range depicted, but the height of the red curve eventually exceeds that of the green and there will be no further intersections beyond such a point. *ep*

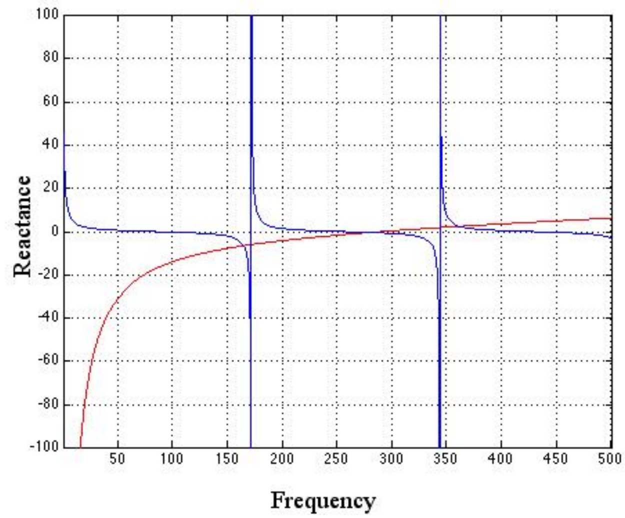

Figure 2. The blue curve is a plot of the cotangent reactance function with the vertical lines indicating the points of discontinuity of this function. The red curve is the loudspeaker reactance curve. Discounting the intersections at the discontinuous jumps, in the depicted frequency range the lowest resonance is at 165 Hz, the middle resonance at 281 Hz is close to the loudspeaker's natural resonance without front loading of 290 Hz, and the upper resonance is at 362 Hz.

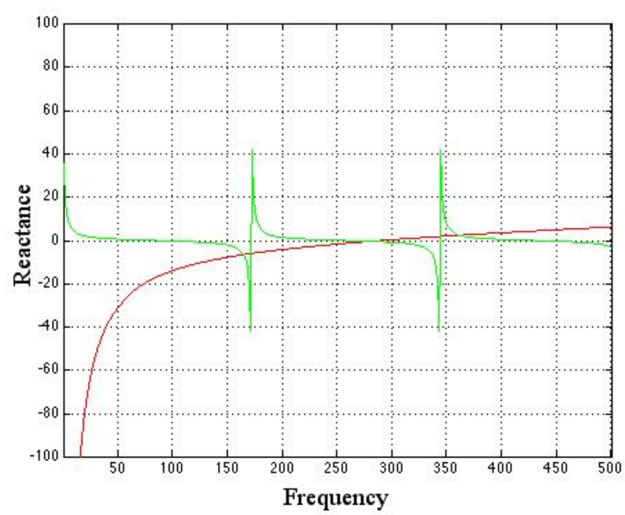

Figure 3. The green curve is the reactance presented by the tube including the effect of air losses. The red curve is the loudspeaker reactance curve. The resonant frequencies indicated here are essentially the same as those of Fig. 2.