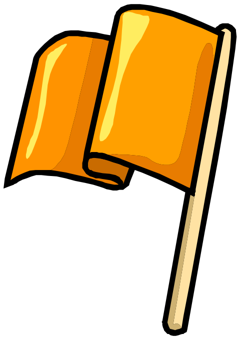
BY DR. EUGENE PATRONIS



What is Waving and Why

The Anatomy of the Wave Equation - Part 5

When a plane wave tube is excited at its origin by a tightly fitting, oscillating piston as illustrated in Fig. 1 the resulting wave motion is that of a plane wave propagating in the direction of increasing z. In this motion, the acoustic pressure and the particle velocity are uniform over the cross-section of the tube and the particle velocity oscillates only in the z-direction. The phase velocity of this plane wave motion is independent of the frequency of excitation. This is not the case for an arbitrary type of excitation nor is it necessarily true when a compression driver excites a plane wave tube as the emerging wave front from such a device may have some curvature. In such instances, one must consider a more general solution to the wave equation consistent with the geometry of the plane wave tube.

In terms of the cylindrical coordinates, (r, θ, and z), that are the simplest ones to employ for the geometry at hand the general solution to the wave equation for acoustic pressure is expressed as a product of three different functions. The first of these functions describes how the acoustic pressure depends on the radial distance from the central or z-axis of the tube. The second function describes how the acoustic pressure varies with the polar angle measured about the central axis. The third function describes how the acoustic pressure varies with regard to both position along the z-axis and with time.

The radial behavior is described by a Bessel function of the first kind of which there are many choices depending upon exactly what mode of wave motion is involved. These Bessel functions are ordered by a subscript m. Bessel functions with orders 0 through 3 are depicted in Fig. 2.

As can be seen from Fig. 2 the Bessel functions of the first kind appear almost as damped sine or cosine functions of the variable x although they are not, as the zero crossings are not periodic. One needs to refer to math tables or computer based math programs to obtain detailed behaviors. The variable x employed in Fig. 2 does not refer to the space variable x but rather to the combination $k_{mn}r$ where r is the radial distance from the z-axis and k_{mn} is the radial wave motion propagation constant. The radial propagation constant k_{mn} requires some extended discussion. First off it has two integral indices m and n. The index m refers to the order of the Bessel function while the index n refers to the order of the position of the variable x in Fig. 2 where the particular Bessel function at hand has zero slope. This zero slope is important because an acceptable solution can only be one for which the radial component of the particle velocity must vanish at the rigid wall of the waveguide and this radial component of particle velocity is proportional to the derivative of the acoustic pressure with respect to the variable r. When r = a, the derivative of the pressure with respect to r,

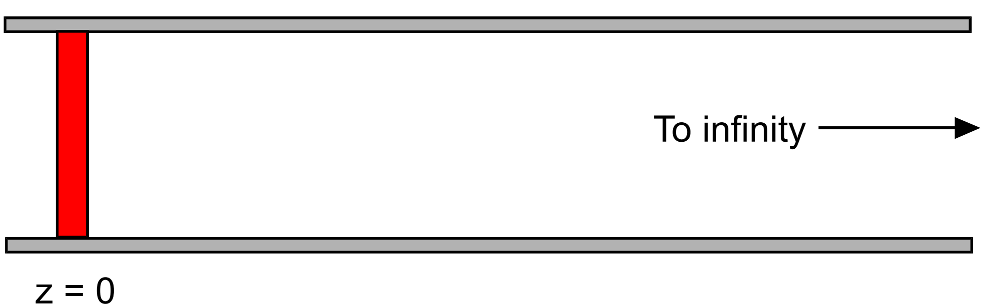


Figure 1. Plane wave tube excited by an oscillating piston at the origin.

that is the slope, must be zero. For example, let m = 1so that we are looking at the blue curve. The first value of x beyond the origin at which the slope of this curve is zero is at the point x = 1.841. This requires then that k_{11} must be 1.841 / a in order for $k_{11}r$ = 1.841 when r becomes equal to a. Similarly, when m = 2 so that we are on the green curve, the first occurrence of zero slope is for x = 3.054 requiring k_{21} be 3.054 / a. The significance of this can be learned from the relationship between the radial propagation constant k_{mn} and the propagation constant along the z-axis that is k_z. This relationship is $k_z = [(\omega / c)^2 - (k_{mn})^2]^{1/2}$. In order to have a propagating mode along the z-axis, k_z must be a real number. This will be true only for those operating frequencies where $(\omega / c)^2 > (k_{mn})^2$ or when $\omega > k_{mn}c$. Consider the mode where m = 1 and n = 1. The frequency below which this mode cannot propagate, that is the cutoff frequency, is given by $f_{11} = 1.841c / (2\pi a)$. Similarly, $f_{21} = 3.054c / (2\pi a)$. Table 1 lists all of the modal cutoff frequencies in the 20 kHz audio band for a one-inch diameter (.0254 meter) plane wave tube. The cutoff frequencies for a two-inch diameter tube are one-half those for a one-inch diameter tube.

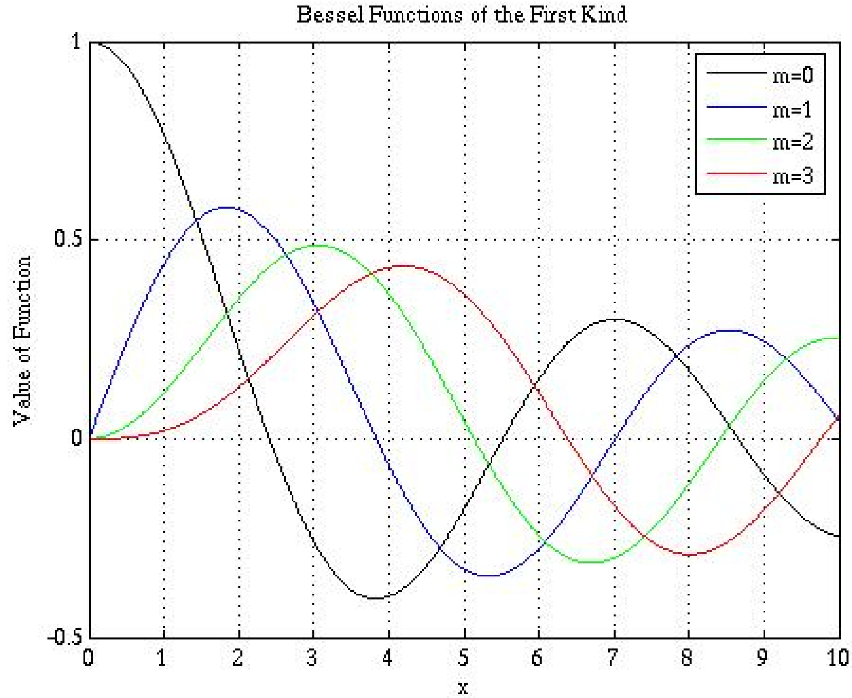


Figure 2. Bessel functions $J_m(x)$ for orders m = 0 through m = 3.

m	n	k_{mn}	f_{mn} (Hz)
1	1	1.841/a	7936
2	1	3.054/a	13166
0	2	3.832/a	16520
3	1	4.20/a	18106

Table 1. Cutoff frequencies in the audio band for a one-inch diameter plane wave tube.

The modes listed in Table 1 are dispersive modes. They propagate at operating frequencies above their respective cutoff frequencies, but do so with a frequency dependent phase velocity. This means that different frequency components of a wideband signal above cutoff propagate with different speeds and wave shapes are not preserved. The phase velocity measured along the z-axis is given by

$$\omega / [(\omega / c)^2 - k_{mn}^2]^{1/2}.$$

Below its respective cutoff frequency, each mode becomes evanescent. This means that the mode does not propagate or transport energy along the z-axis but rather its pressure contribution attenuates exponentially with distance from the origin.

A reasonable question to ask at this point is, “What does the solution look like with all of these added complications?” The answer is a sum over all indices that can contribute for a given operating frequency or range of operating frequencies of pressure terms of the following structure

$$p_{mn}(r, \theta, z, t) = A_{mn}J_m(k_{mn}r)\cos(m\theta)\cos(\omega t - k_z z).$$

In the above equation A_{mn} are amplitude factors determined by the conditions at the exciting source. These amplitude factors differ depending on the values of m and n, namely on the particular mode involved. All of these non-planar modes have non-uniform pressures as well as polarities over the cross section of the waveguide. The next question should be, “Where is our familiar plane wave solution in all of this?” The answer again lies in a further examination of Fig. 2. Notice that when m = 0, the function J_0 has zero slope when x = 0, that is right at the origin. For this case, not only is m = 0 but n and k_{00} are zero as well while $J_0(k_{00}r)$

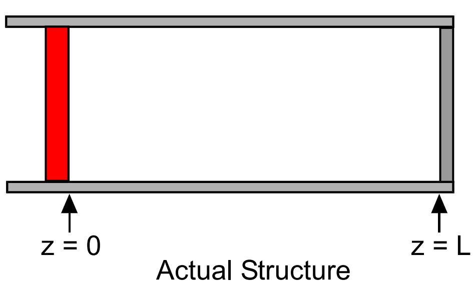
has the value of one independent of r. As m = 0, cos(mθ) is unity independent of the angle θ and the solution becomes the familiar

$$p(z, t) = A\cos(\omega t - k_z z)$$

with A being the pressure amplitude at the face of the piston that is uniform over the cross section of the tube. Furthermore, as k_{00} is zero, the phase velocity is a constant value c at all frequencies.

Next we turn our attention to tubes of finite length that are excited only with plane waves but have arbitrary terminations. A good starting point is a tube with length L excited by a piston as in our original case but which is terminated by a rigid barrier at its receiving end. This situation is depicted in the upper half of Fig. 3.

As the piston begins to move in the actual structure a plane wave propagates in the positive z-direction reaching the rigid barrier after a time lapse of L / c. As the barrier is rigid, there can be no particle displacement or velocity at the barrier at any time. The wave is reflected and then travels in the negative z-direction back to the source where it is again reflected but now by the moving piston. This process continues to repeat itself over and over and after numerous transits back and forth arrives at a steady state condition with the boundary conditions at the piston matching those of the piston's motion and those at the barrier corresponding to zero particle displacement as well as velocity. In the region $0 \le z \le L$ we ultimately have the superposition of two waves traveling in opposite directions. The wave equation is linear so the principle of superposition is applicable in arriving at a solution for a particular case. Furthermore, the solution to the wave equation that satisfies the given boundary conditions is unique so we can treat the problem by analyzing the equivalent structure in the lower half of Fig. 3 that is based on the method of images. In this technique the red piston is the actual source and the blue piston is the image source that is located just as far to the right of the barrier as the actual source is located to the left of the barrier. When the red piston displaces to the right, the blue piston displaces similarly to the left. In the equivalent structure there is no barrier at z = L although its position is indicated in the drawing. In the active interval $0 \le z \le L$ in which our solution will apply we sum the individual waves generated by the real source and the image source. When the red piston moves to the right the air in front of it is compressed so it produces a pressure wave given by $p_r = p_m\cos(\omega t - kz)$ where p_m is the pressure amplitude and k is the propagation constant along the z-axis. Here we have dropped the subscript on the propagation constant as we are dealing only with a plane wave. Similarly, when the image piston moves to the left it produces a pressure wave $p_b = p_m\cos(\omega t + kz')$. The situation with regard to the particle velocity is decidedly different however. The particle velocity wave generated by the red piston is $u_r = u_m\cos(\omega t - kz)$, but when the blue piston moves to the left, the air particles are moving in the negative direction so $u_b = -u_m\cos(\omega t + kz')$. Furthermore, $z' = z - 2L$ so in the active interval $0 \le z \le L$ the total acoustic pressure is given by $p = p_m\cos(\omega t - kz) + p_m\cos(\omega t + kz - 2kL)$ while the total



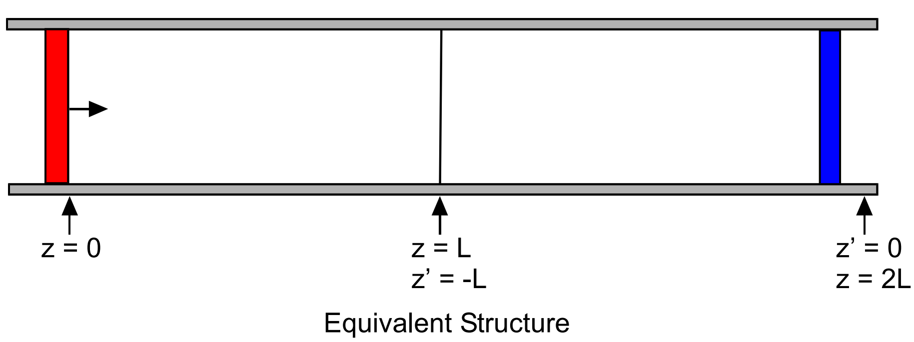


Figure 3. Tube terminated by a rigid barrier.

particle velocity is given by $u = u_m\cos(\omega t - kz) - u_m\cos(\omega t + kz - 2kL)$. These composite expressions describe standing waves. Our traveling waves in opposite directions have combined to form standing waves of both acoustic pressure and particle velocity. Now at the barrier where $z = L$, the acoustic pressure is $p = 2p_m\cos(\omega t - kL)$. This means that the pressure amplitude is doubled signifying that a normally incident pressure wave is reflected in phase at a rigid barrier. On the other hand, the particle velocity at the barrier as given by $u = u_m\cos(\omega t - kL) - u_m\cos(\omega t - kL)$ is identically zero at all times indicating that a normally incident particle velocity wave is reflected with a change of polarity or a phase shift of π radians at a rigid barrier. The important question is, "What are the conditions at the origin where the red piston is located?" Upon setting $z = 0$ in the general equations we find that the total acoustic pressure at the origin is now $p = p_m\cos(\omega t) + p_m\cos(\omega t - 2kL)$. Now it is important to remember at this point that all angles are expressed in radians. Suppose that the length L is exactly $\lambda / 4$ at the operating frequency of the piston. Upon remembering that $k = 2\pi / \lambda$, then $2kL = 2(2\pi / \lambda)(\lambda / 4) = \pi$. When this is the case, the two pressure terms differ in phase by π and their sum is zero at all times t. This is a resonant condition. The acoustic pressure being identically zero means that the piston motion is completely unimpeded. This will be true also when L is any odd integral multiple of $\lambda / 4$. This ideal is never exactly achieved in practice because there are always some very small viscous losses at the walls of the tube and the amplitude of the blue piston motion is always slightly less than that of the red piston.

The conditions that exist at the exciting piston for a tube of arbitrary length L terminated in a rigid barrier are usually studied by means of the mechanical impedance presented to the piston as a result of its interaction with the air at the origin. This mechanical impedance is the ratio of the complex force acting on the air at the piston face to the complex particle velocity of the air at the piston face. The complex force is the acoustic pressure at the origin expressed as a complex exponential or phasor multiplied by the cross sectional area of the tube. The complex particle velocity is the complex exponential statement of the particle velocity also at the origin. Complex exponentials and phasors are described in detail in Chapter one of *Sound System Engineering*, Third Edition. The mechanical impedance then is calculated from

$$Z_m = \frac{p_m S}{u_m} \frac{e^{j\omega t} + e^{j(\omega t - 2kL)}}{e^{j\omega t} - e^{j(\omega t - 2kL)}}$$

This can be simplified by factoring and canceling common terms in both the numerator and denominator to yield,

$$Z_m = \frac{p_m S}{u_m} \frac{1 + e^{-j2kL}}{1 - e^{-j2kL}}$$

$$Z_m = \frac{p_m S}{u_m} \frac{e^{-jkL}}{e^{-jkL}} \frac{e^{jkL} + e^{-jkL}}{e^{jkL} - e^{-jkL}}$$

$$Z_m = \frac{p_m S}{u_m} \frac{\cos(kL)}{j\sin(kL)}$$

$$Z_m = -j\frac{p_m S}{u_m} \cot(kL)$$

The conclusion is that the mechanical impedance presented to the piston is a negative reactance for positive values of the cotangent as indicated by the negative j in the final statement. If there were no viscous losses this impedance would be zero at the resonant condition where L equals odd integral multiples of $\lambda / 4$ and would be infinite for the anti-resonant condition where L equals integral multiples of $\lambda / 2$. The fact that the mechanical impedance is purely reactive means that once steady state conditions are reached the average power supplied by the piston becomes zero in the ideal case. It also means that the acoustic pressure and particle velocity in the standing wave differ in phase by $\pi / 2$ radians or 90°.

Next time we will deal with a tube having an open end termination and explore the tube's influence on the motion of a small loudspeaker used to excite the tube. *ep*