BY DR. EUGENE PATRONIS

What is *Waving* and Why?

The Anatomy of the Wave Equation - Part 1

This is the first of what hopefully will be a continuing series of articles dealing with the fundamental physics of sound waves. Unlike electromagnetic waves that can exist in a vacuum as well as in material substances, sound waves are mechanical waves and require a material medium in which to exist. The medium may be either a solid such as a bar of steel or a fluid such as water or air. Fluids are distinguished from solids in that a fluid will assume the shape of the container in which it is placed. If the fluid in question is a liquid of small volume, however, it will not occupy all of the space provided by a container of larger interior volume whereas a gaseous fluid will occupy all of the interior space provided by the container. In the process of doing so, the pressure, temperature, and the energy content of the gas must undergo adjustments to make this possible. The thrust of this series of articles is to learn in a fundamental way that wave properties such as speed, dispersion, momentum transfer, energy transport, and guidance depend upon both physical law and the properties of the host medium. Along the way we will encounter all of the familiar wave properties such as interference, diffraction, and refraction as well as phase and group velocity. It is a fact that air borne sound plays a dominant role in sound reinforcement and room acoustics so we will begin our study with air considered to be the supporting medium for our initial treatment of sound waves.

Air is a mixture of several different gaseous components with the principal ones being displayed in Table 1 as molecular fractions of the total composition. The numbers displayed as decimal fractions are for dry air. Normal air also contains water vapor. This does not appear in the table, as it is a varying quantity depending upon the weather conditions. The major acoustic influence of the moisture content of air is that of a frequency dependent conversion of sound energy into heat. This process will be described at the appropriate time.

Gas	Symbol	Fraction
Nitrogen	N_2	0.78040
Oxygen	O_2	0.20946
Argon	Ar	0.00934
Carbon dioxide	CO_2	0.00038

Table 1. Principal components of dry air at sea level.

Note that if you sum the fraction column the result will be slightly less than one. The reason for this is that dry air also contains trace amounts of hydrogen, helium, neon, krypton, xenon, radon, and methane. The fractions presented in the table are the molecular fractions rather than the mass fractions. This means that if you take a sample of dry air at sea level 78.040 % of the molecules in the sample will be diatomic molecules of nitrogen while 20.946 % will be diatomic molecules of oxygen, etc. We could construct a similar table where fractions of the total mass rather than fractions of the total number of molecules represent the various components. The mass fractions would be different because, for example, molecules of oxygen have greater mass than do molecules of nitrogen. In discussing molecular masses associated with a sample of substance that you might encounter in everyday life, i. e., a macroscopic sample we usually refer to a mole of the substance. The mole is one of the seven SI base units and is called quantity of substance. The mole has the value $6.02(10^{23})$ which is Avogadro's number. You can have a mole of anything. A mole of dollars would be $\$6.02(10^{23})$. With that amount of money you could give to each person on earth an amount equal to the national debt of the US and never miss it! A mole of dry air has a mass of 0.02898 kilogram. This is called the molar mass of dry air. The molar mass will be represented in this article by M.

Reference conditions for a standard atmosphere are usually taken as the pressure at sea level with a tempera-

ture of 0 °C. This temperature on the Celsius or centigrade scale corresponds to 273.15 K on the absolute or Kelvin temperature scale. You should note that we did not write °K because K alone stands for degrees on the absolute temperature scale. Note also that the degree increments on each scale are the same size so any temperature reading on the Celsius scale can be converted to absolute simply by adding 273.15. The standard atmosphere has a static or undisturbed pressure of $1.01325(10^5)$ Pascals and a density of 1.293 kg•m^{-3}. A sound with a SPL of 94 dB corresponds to an rms acoustic pressure of 1 Pascal. This moderately loud sound, which represents a disturbance away from static conditions, perturbs the atmospheric pressure less than one part in 10^5. Accompanying the pressure disturbance there are also disturbances in the local air's density and temperature. In order to understand this we must consult what is called the equation of state of a gas. Furthermore the sound source feeds acoustic energy into the air. In order to get a handle on this we must consult the first law of thermodynamics.

The equation of state of an ideal gas as given by the ideal gas law says

$$PV = nRT$$

where P is the total pressure exerted by the gas on the walls of the containing vessel, V is the interior volume of the container, T is the absolute temperature of the gas, n is the number of moles of the gas in the container, and R is the universal gas constant. The value of R is 8.3145 Joule•mole^{-1}•K^{-1}. An ideal gas would be one in which collisions between individual molecules of the gas as well as collisions with the walls of the container would be perfectly elastic, in which both momentum and kinetic energy are conserved in the collision processes. Real gases do not obey the ideal gas law under all possible conditions. The ideal gas law was determined by experimentally studying the behavior of real gases as a function of the gas density. With sufficiently low densities all real gases were found to follow the same equation of state that has become to be known as the ideal gas law. The behavior of air in the temperature and pressure ranges that we normally encounter in everyday life follows the ideal gas law with little error. We should note at this point that the ideal gas law could be expressed by using the gas density rather than the gas volume as a variable of interest. Let the total mass of our sample of gas be represented by m. Then the number of moles in our sample of gas would become m/M and we can write the following sequence of equations:

$$PV = nRT$$

$$PV = \frac{m}{M}RT$$

$$P = \frac{m}{V}\frac{R}{M}T$$

$$P = \rho\frac{R}{M}T$$

In the last equation the mass per unit volume or density of the gas is represented by the Greek letter rho.

The first law of thermodynamics in simple terms states that the change in the internal energy of a physical system is equal to the heat energy added to the system less the work done by the system. If we let Q represent the added heat energy and W represent the work done, then $\Delta U = Q-W$, where ΔU stands for the change in internal energy. In the case of a system composed of an ideal gas, the internal energy is associated only with the kinetic energies of the molecules composing the gas. The individual molecules in such a gas undergo random motions throughout the volume of the gas and have speeds that can change from moment to moment as a result of collisions with other molecules or with the walls of the containing vessel. Monatomic molecules such as Ar can only have kinetic energies associated with translations in the three perpendicular spatial directions. We say such molecules have three degrees of freedom. Diatomic molecules such as N_2 and O_2 in addition to translation can have kinetic energies associated with rotations about two perpendicular axes. Such molecules are said to have five degrees of freedom. Finally, polyatomic molecules such as CO_2 can potentially perform distinct rotations about three mutually perpendicular axes and have six degrees of freedom. Kinetic theory tells us that for each molecule in the gas there is on the average a kinetic energy that is proportional to kT where k is Boltzman's constant with k being equal to R divided by Avogadro's number or $1.38(10^{-23})$ Joule•K^{-1}. The significance of this is that the internal energy of an ideal gas or a real gas that behaves as an ideal gas is directly proportional to the absolute temperature alone. Regardless of the complexity of a molecule's structure, each molecule in the gas has an average translational kinetic energy of 3/2 kT. Knowledge of a molecule's average kinetic energy allows the calculation of the root mean square molecular speed. The formula for this calculation is

$$v_{rms} = \sqrt{\frac{3RT}{M}} \, .$$

If you apply this formula to air at standard conditions v_{rms} will be found to be 485 m•sec^{-1}.

As a result of many experiments it has been determined there are two types of air compression and expansion processes associated with sound waves. The first and simplest of these occurs normally at ultrasonic frequencies, well above the audible spectrum but can also occur in a sealed box loudspeaker enclosure which is completely filled (except for the volume occupied by the speaker) with loosely packed fiberglass. This is the isothermal process. The second process is known as the adiabatic process and is applicable in free air throughout the audible spectrum and beyond. An isothermal process is one in which the gas temperature and hence its internal energy remains constant. An inspection of the ideal gas law for a fixed quantity of gas and a fixed temperature reveals that the isothermal process is described by $PV = C_T$ where C_T is just a constant. For one mole of air at standard conditions $C_T = 2270$ Joule. An adiabatic process is one in which heat energy is neither added nor subtracted. For an adiabatic process not only must the pressure-volume relationship follow the ideal gas law it must also satisfy $PV^\gamma = C_Q$ where C_Q is a constant that depends on the state of the gas and γ is a constant for the particular mixture composing the gas. For air γ has the value of 1.402 and is dimensionless whereas for one mole of air at standard conditions C_Q has the value 493 with the strange dimensions of Pascals•(m^3)$^\gamma$. A mole of any ideal gas under the standard conditions of sea level atmospheric pressure and a temperature of 273.15 K occupies a volume of 0.0224 m^3 regardless of the type of expansion or compression process that may occur. This is illustrated in Fig. 1 for one mole of air where a comparison is made between plots of the pressure-volume relationship for both an isothermal and an adiabatic process.

At the intersection point of the two curves, the gas pressures, volumes, and temperatures are the same. In the isothermal process, the first law of thermodynamics tells us that when air does an amount of work W against an outside agency in expanding beyond the intersection point, an amount of heat Q equal to W must be added to the air in order to maintain the temperature T at a fixed value. Conversely when an outside agency compresses the gas and thus does work on the gas while reducing its volume, a corresponding amount of heat energy must be removed from the gas in order to maintain a constant T. With a constant temperature there is no change in the internal energy of the gas in either circumstance. In the adiabatic process, however, no heat energy is added or removed so when the gas does work during expansion, the gas itself must supply this energy. As a consequence
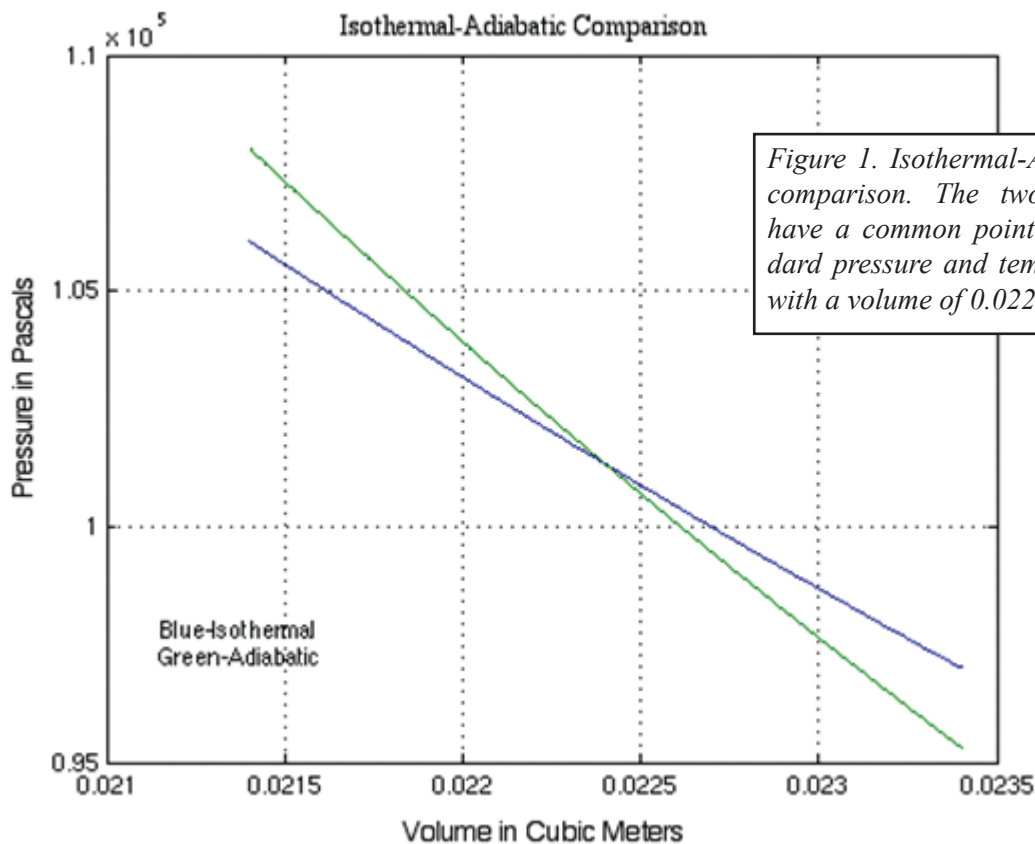

Figure 1. Isothermal-Adiabatic comparison. The two curves have a common point at standard pressure and temperature with a volume of 0.0224 m^3.

the internal energy of the gas decreases and the gas temperature drops. Conversely, of course, when an outside agency compresses the gas adiabatically, work is done on the gas rather than by the gas and the internal energy of the gas increases by an amount equal to the work done. This manifests itself as an increase in gas temperature.

We have mentioned the work done on or by a gas several times without divulging how the work is determined. We hasten now to remove that omission. Suppose the gas is air and it is contained in a cylinder that has a tightly fitted piston upon which we can exert a sufficient force to move the piston inward so as to compress the gas adiabatically. Alternatively, we can relax our force on the piston somewhat and allow the gas to expand adiabatically and thus push the piston outward. In order to do this the entire apparatus must be thermally insulated from the external environment so that heat energy cannot leak in or out during the process. The formal definition of the work done by the gas in the process of the volume changing from an initial value V_i to a final value V_f is given by the following integral equation.

$$W = \int_{V_i}^{V_f} P dV$$

This integral has a geometrical interpretation. It actually calculates the area that lies beneath the plot of pressure versus volume between the limits of V_i and V_f. When the final value of volume is greater than the initial value the numerical value of the calculation is positive indicating that the gas has done work against the piston while expanding. If the opposite is true, final volume less than initial volume, the numerical value is negative indicating that the piston has done work on the gas by compressing it. We should also observe that if we multiply the dimensions of pressure by the dimensions of volume we obtain the dimensions of energy thusly, Newton•m^{-2}•m^3 = Newton•m. A Newton•m is of course a Joule. An example of this is displayed in Fig. 2 where a mole of air is compressed from an initial volume of 0.022 m^3 to a smaller volume of 0.018m^3.

The value of the integral is –478 Joules with the minus sign indicating that work was done on the gas. In other words, the agency pushing the piston did a work of 478 Joules in compressing the gas. This amount of energy is potential energy stored in the compressed gas because the gas could do this amount of work on the piston in expanding back to its original state. You can visually do a rough calculation of the area under the graph by noting that each complete crosshatched block has an area of 0.002•20,000 or 40 Joules and there are 11
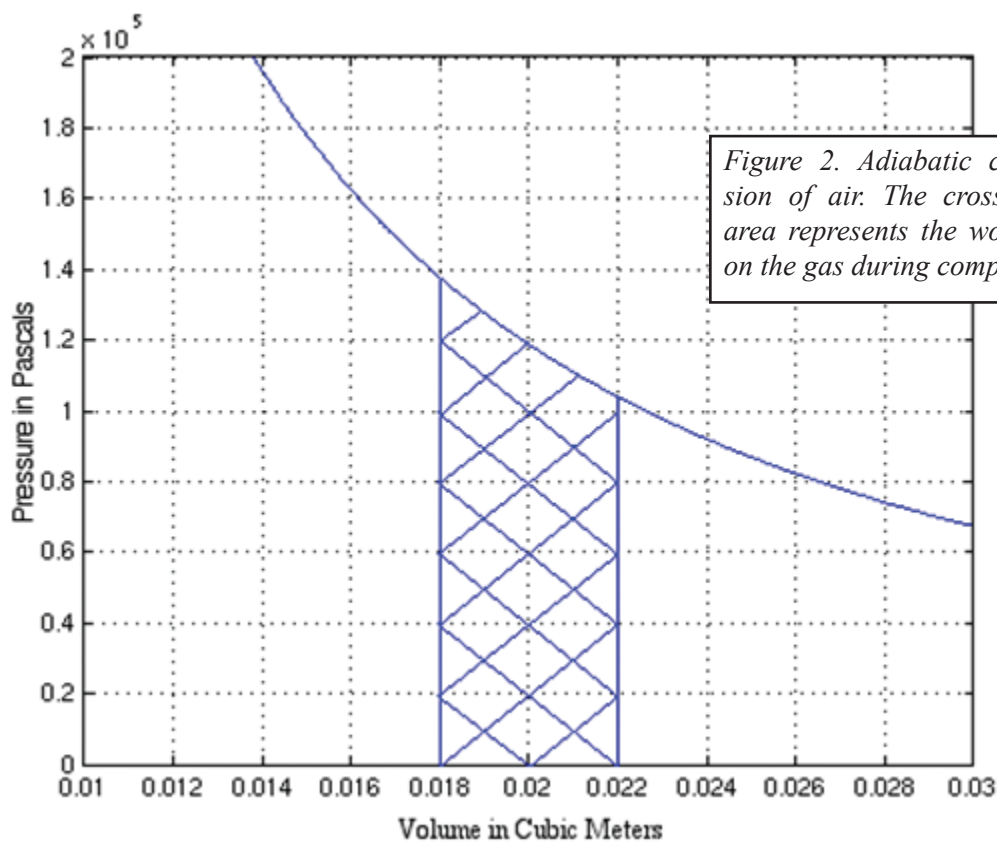

Figure 2. Adiabatic compression of air. The crosshatched area represents the work done on the gas during compression.

complete blocks and a slightly less than complete partial block. The eleven complete blocks correspond to 440 Joules.

Finally, we are ready for an acoustical calculation of some importance. We previously mentioned that an adiabatic process was described by $PV^\gamma = C_Q$ where the constant C_Q depends on the quantity of air involved. An alternative description of an adiabatic process for air in the atmosphere that is independent of the amount of air involved is

$$P = \left(\frac{\rho}{\rho_0}\right)^\gamma P_0.$$

In this equation P is the total air pressure when the air density is ρ and P_0, ρ_0 are the values of the air pressure and density under standard conditions. Fig. 3 illustrates this behavior.

If we examined the curve in the vicinity of the marker on a greatly magnified scale the curve would appear to be a straight line and we could determine its slope graphically. This slope describes the ratio of a small change in pressure to a small change in density. Alternatively, we could employ the methods of differential calculus and precisely determine a value for the slope. Calling this slope c^2, the slope of the curve at the position of the marker is

$$c^2 = \gamma \frac{P_0}{\rho_0}.$$

Upon inserting the values for standard air pressure and density c^2 is found to be 109866.71 with the dimensions of Newton•m•kg^{-1} = m^2•sec^{-2}. The obvious next step is to extract the square root to find c = 331.46 m•sec^{-1}. This result should be familiar as it is the speed of sound in air at a temperature of 273.15 K! This of course is interesting and the explanation for it will be forthcoming. It is not, however, the calculation that is to close this article. Let us turn our attention to a cubic centimeter of air under standard conditions. A cubic centimeter is 10^{-6} cubic meter so our sample contains a mass of 1.293(10^{-6}) kg or one milligram of air. Let our air sample be in contact with some vibrating object such that the sample is momentarily compressed by a very small amount. The one-milligram of air now occupies a smaller volume so its density has increased and this is accompanied by an increase in the air pressure in the sample itself. Let the new pressure be P = 101326 Pascal. The undisturbed pressure was P_0 = 101325 Pascal. What we call acoustic pressure, p, is the difference between these two numbers so p = P−P_0 = 1 Pascal. Now we can use the slope on Fig. 3 to calculate the change in density from p = $c^2(\rho−\rho_0)$.
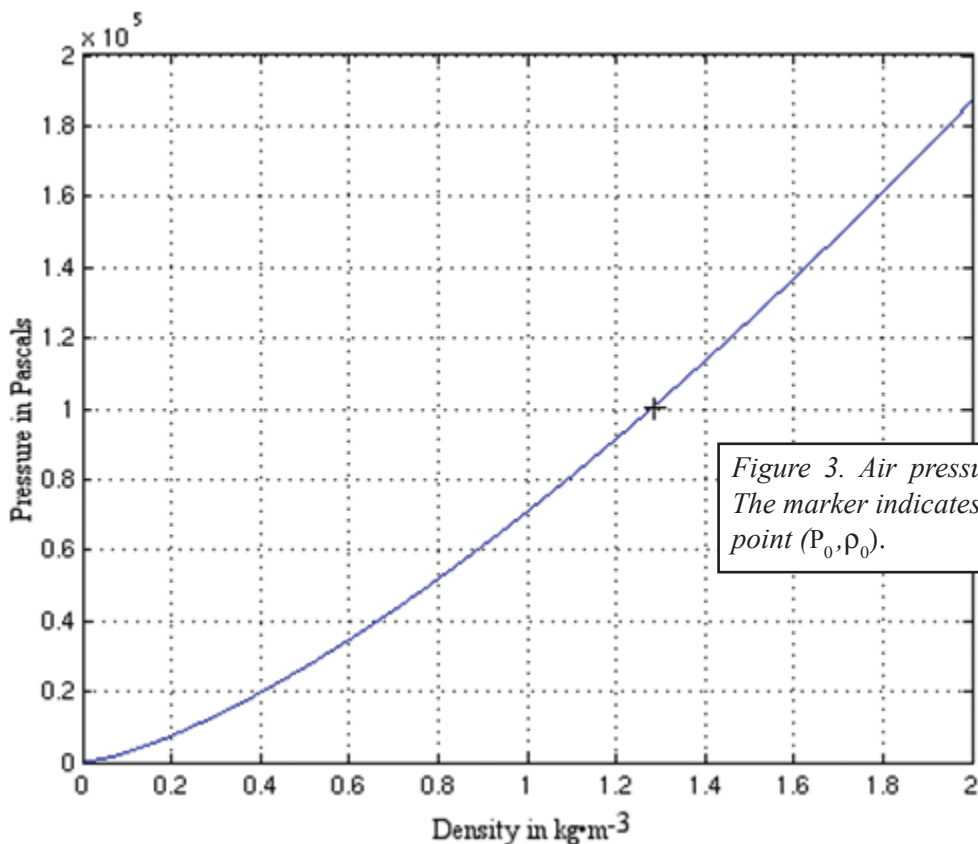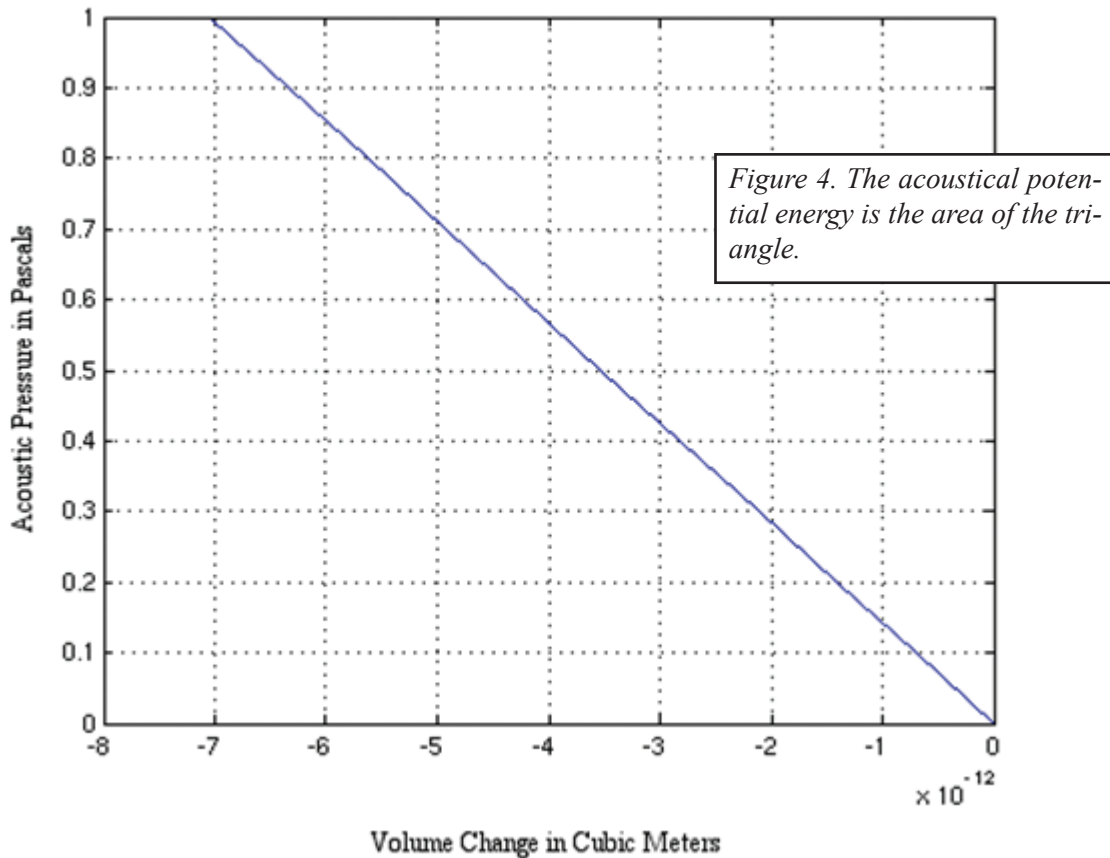

Figure 3. Air pressure versus density. The marker indicates the position of the point (P_0, ρ_0).

Acoustic Pressure in Pascals (y-axis)

Volume Change in Cubic Meters (x-axis) $\times 10^{-12}$

Remember the slope is pressure change divided by density change. We can solve for ρ to find $\rho = 1.293009102$ kg•m^{-3}. We started with a mass of air of $1.293(10^{-6})$ kg that was contained in an initial volume V_0 of 10^{-6} m^3. We still have the same mass of air but now in a smaller volume V. Making use of this fact we can write

$\rho_0 V_0 = \rho V$. Solving this for V we find that $V = 9.99929606(10^{-7})$ m^3. The change in volume in the compression process is then $V-V_0 = -7.039393602(10^{-12})$ m^3. The result of this is that there is now energy stored in our compressed sample of air. A portion of this stored energy is called acoustical potential energy and is represented by the work that can be performed by the <u>acoustic pressure</u> as the sample expands back to its original volume against the external pressure of the surrounding air. This work can be determined from Fig. 4 in which the acoustic pressure is plotted versus the size of the volume change.

The work that can be performed by the acoustic pressure as the sample expands back to normal size is the area included in the triangle and in this case is one-half the altitude times the base or $0.5•7.039393602(10^{-12})$ Pascal•m$^3 = 3.5197(10^{-12})$ Joule. We can calculate this directly using the equation for acoustical potential energy that is obtained through the full use of calculus.

$$E_P = \frac{1}{2}\left(\frac{p^2}{\rho_0 c^2}\right)V_0$$

The equation given above is an important result that we will make full use of in the next episode of this saga where we will see the origin of the wave equation and use it to explain what goes on in plane wave tubes for a variety of terminations. We will encounter traveling waves and standing waves as well as acoustical energy transport. *ep*

"There is hardly anything in the world that some man can't make a little worse and sell a little cheaper, and the people who consider price only are this man's lawful prey."

John Ruskin - 1850