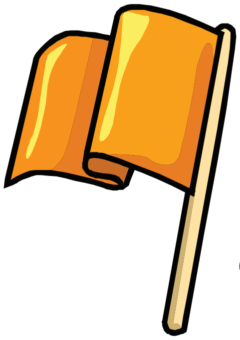
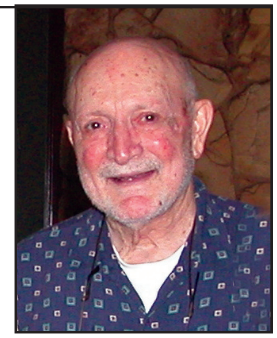
BY DR. EUGENE PATRONIS



What is Waving and Why?



The Anatomy of the Wave Equation - Part 3

At the conclusion of the second article in this series we had the plane wave equation in hand and had learned the properties that must be exhibited by a mathematical function if it were to be a solution to the wave equation for a given set of physical circumstances. Now it is time to apply what we have learned towards the study of the wave motion in a plane wave tube that is excited by an oscillating piston as depicted in Fig. 1.

For simplicity, we will consider the circumstance where the piston, depicted in red, has been forced to undergo oscillatory motion for some time by a mechanism not shown in the figure and continues to do so as we study the problem. We will start measuring time from the instant when the right face of the piston is just passing $z = 0$ and is moving to the right such that the piston's displacement from $z = 0$ is described by $\xi = \xi_m \sin(\omega t)$ where $\xi\mu$ is the amplitude of the piston displacement, $\omega = 2\pi f$ with f being the frequency of oscillation in Hz, and t is the time. Now we want to find a solution to the wave equation for air particle displacement in the tube that satisfies these conditions. The tube is infinitely long so there can be no reflections from the receiving end. As a consequence, we need only a solution that describes a wave traveling to the right. The air in contact with the piston undergoes the same motion as does the piston itself so we propose as a solution an expression that duplicates the piston motion when we let $z = 0$

$$\xi = \xi_m \sin(\omega\, t - kz)$$

We also learned last time that in order for a solution to legitimately describe a plane wave propagating in the direction of increasing z that the space and time variables must appear in the form $(ct - z)$. We can easily show that our proposed solution satisfies this requirement as follows. The quantity k is called the propagation constant and is defined as $k = 2\pi / \lambda$ where λ is the wavelength. Now as ω is $2\pi f$ and ω / k is $\lambda f = c$, then if we factor k out of our parenthesis in our proposed solution, the solution will take the form

$$\xi = \xi_m \sin\left[k(ct - z)\right]$$

Since the two expressions for the particle displacement are equivalent we may use either form to suit our convenience. Next, it is necessary to show that our proposed solution when substituted into the wave equation produces an identity. In accomplishing this it is necessary to take partial derivatives of our proposed solution first with respect to z and then with respect to t. The first partial derivative with respect to z finds the slope of

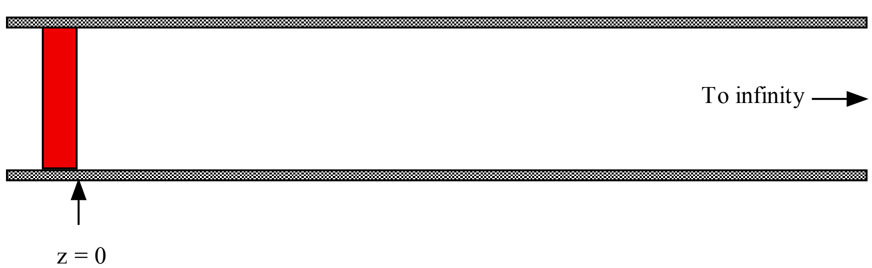


Figure 1. Plane wave tube that is fitted with an oscillating piston.

the solution when z is allowed to change while t is held at a constant value. Similarly, the second partial derivative with respect to z finds the slope of the slope curve while z is allowed to change with t being held constant. The process is then repeated except now t is allowed to change while z is held at a constant value. The results are found to be

$$\frac{\partial^2 \xi}{\partial z^2} = -k^2 \xi_m \sin(\omega t - kz)$$

$$\frac{\partial^2 \xi}{\partial t^2} = -\omega^2 \xi_m \sin(\omega t - kz)$$

The wave equation tells us to divide the second partial derivative with respect to t by c^2 and equate the result to the second partial derivative with respect z. If an identity results from this action then our proposed solution does indeed satisfy the wave equation. Upon dividing the second equation immediately above by c^2 and equating it to the first equation immediately above we obtain

$$-\frac{\omega^2}{c^2} \xi_m \sin(\omega t - kz) = -k^2 \xi_m \sin(\omega t - kz)$$

This is indeed an identity because $k = 2\pi/\lambda = 2\pi f / c = \omega / c$. Finally, when we let z = 0, our air particle displacement agrees with the piston motion at all times t including t = 0. Therefore our proposed solution satisfies all of the requirements necessary to be the one and only solution to the problem.

What about the particle velocity and the acoustic pressure? We obtain the particle velocity from the partial derivative of the particle displacement with respect to t.

$$u = \frac{\partial \xi}{\partial t} = \omega \xi_m \cos(\omega t - kz)$$

The acoustic pressure is obtained from $-\rho_0 c^2 \frac{\partial \xi}{\partial z}$. We learned this in the second article of this series of articles.

$$p = \rho_0 c^2 k \xi_m \cos(\omega t - kz)$$

It is important to note that if we divide the acoustic pressure expression by that of the particle velocity we obtain a quantity called the specific acoustic impedance of air for plane waves namely,

$$\frac{p}{u} = Z_s = \frac{\rho_0 c^2 k}{\omega} = \rho_0 c$$

Here the capital letter Z represents impedance rather than the spatial coordinate and the subscript s stands for specific. The specific acoustic impedance of air for plane waves is a real number denoting the fact that the acoustic pressure and particle velocity are in phase. The dimensions of Z_s are $kg \cdot m^{-2} \cdot sec^{-1}$. This combination is called a Rayl in honor of Lord Rayleigh who was a pioneer in the study of sound and acoustics.

Now we will put the theory into practice with a realistic numerical example. Let the frequency of oscillation of the piston be 1000 Hz and let its displacement amplitude be 10^{-6} meter. Let the static air pressure be the sea level value but let the temperature be a comfortable 70° F. This corresponds to 21.11° C or 294.26 K. The static air density is inversely proportional to the absolute temperature so then $\rho_0 = 1.293(\ 273.15 / 294.26\ ) = 1.20$ $kg \cdot m^{-3}$. The speed of sound is directly proportional to the square root of the absolute temperature so $c = 331.46(\ 294.26 / 273.15\ )^{0.5} = 344\ m \cdot sec^{-1}$. The air particle displacement amplitude matches that of the piston so $\xi_m = 10^{-6}$ m. The angular frequency $\omega = 2\pi f = 6{,}283$ radians / sec. The propagation constant $k = \omega/c = 18.265\ m^{-1}$. The velocity amplitude is $u_m = ck\xi_m = \omega\xi_m = 6.283(10^{-3})$ $m \cdot sec^{-1}$. The acoustic pressure amplitude $p_m = \rho_0 c u_m =$ 2.5937 Pascal. The rms pressure for sinusoidal time dependence is the amplitude multiplied by 0.7071 and is 1.834 Pascal. This corresponds to a SPL of 99.25 dB. The wavelength $\lambda = c/f = 0.344$ meter. Our solutions for the acoustic variables expressed as functions of both position and time are then

$$\xi = 10^{-6} m \bullet \sin\left(\frac{6{,}283}{sec} t - \frac{18.265}{m} z\right)$$

$$u = 6.283(10^{-3}) m \bullet sec^{-1} \bullet \cos\left(\frac{6{,}283}{sec} t - \frac{18.265}{m} z\right)$$

$$p = 2.5937\ Pa \bullet \cos\left(\frac{6{,}283}{sec} t - \frac{18.265}{m} z\right)$$

Given that the piston has been oscillating for some time, Fig. 2 depicts the acoustic pressure wave propagation along a one-wavelength interval of the z-axis versus elapsed time commencing from the instant when the piston is located at z = 0 and is moving in the positive z direction.

Now for a pop quiz! If we were to construct a ninth entry to Fig. 2 corresponding to t =. 001 second, how would it look? Hint: .001 second corresponds to the period of the motion of the piston and is equal to the time required for the pressure wave to travel a distance of one wavelength along the z-axis. This being the case, the ninth entry would look exactly like the first. Furthermore, a slight modification of Fig. 2 would allow it to describe the particle velocity as well. This modification would involve only a change of scale and label for the vertical axes as the particle velocity is in phase with the acoustic pressure for a plane wave in air.

Now that we have established the behavior of the sound wave in the tube it is appropriate to consider what the piston's motion must accomplish to bring about this behavior. The piston of course is displacing the air adjacent to its right hand face. The piston must exert a force on the air in order to displace it and this requires that the piston perform work on the air. The force, F, exerted by the piston at any instant is the acoustic pressure at z = 0 multiplied by the cross-sectional area of the tube namely, S.

$$F = Sp_m \cos(\omega\ t)$$

The rate at which the piston is performing work on the air is the instantaneous power or P and is obtained by multiplying the applied force by the rate of displacement at z = 0. The rate of displacement at z = 0 is just the particle velocity at the origin so

$$P = Sp_m \cos(\omega\ t) \bullet u_m \cos(\omega\ t) = Sp_m u_m \cos^2(\omega\ t)$$

Fig. 3 is a plot of this result for one period of the piston motion using the values from our numerical example when applied to a plane wave tube having an inner diameter of 1 inch or 0.0254 m.

Fig. 3 displays two items of interest: the plot of the instantaneous power versus time and the area beneath the power curve that is shaded blue. Since the average value of the $\cos^2$ over one period is 1/2 the area under the curve is 1/2 $P_m \cdot 0.001$ sec. For our example this would be 4.1288(10^{-9}) Joule. This area accounts for the total acoustic energy delivered to the sound wave during one period of the piston's motion. One can reasonably inquire as to where this energy resides in the sound wave. The acoustical energy associated with a plane wave appears in two forms. First there is acoustic kinetic energy associated with the motion of the air particles themselves and then there is acoustic potential energy associated with the existence of acoustic pressure. We encountered the concept of acoustic potential energy in part one of this series of articles. This acoustic energy is not localized at a point but rather is distributed throughout the volume occupied by the wave with an energy density that varies as a function of position and time. If we let $\mathbf{e}$ represent the total acoustic energy density while $\mathbf{e}_k$ and $\mathbf{e}_p$ represent the kinetic and potential energy densities,

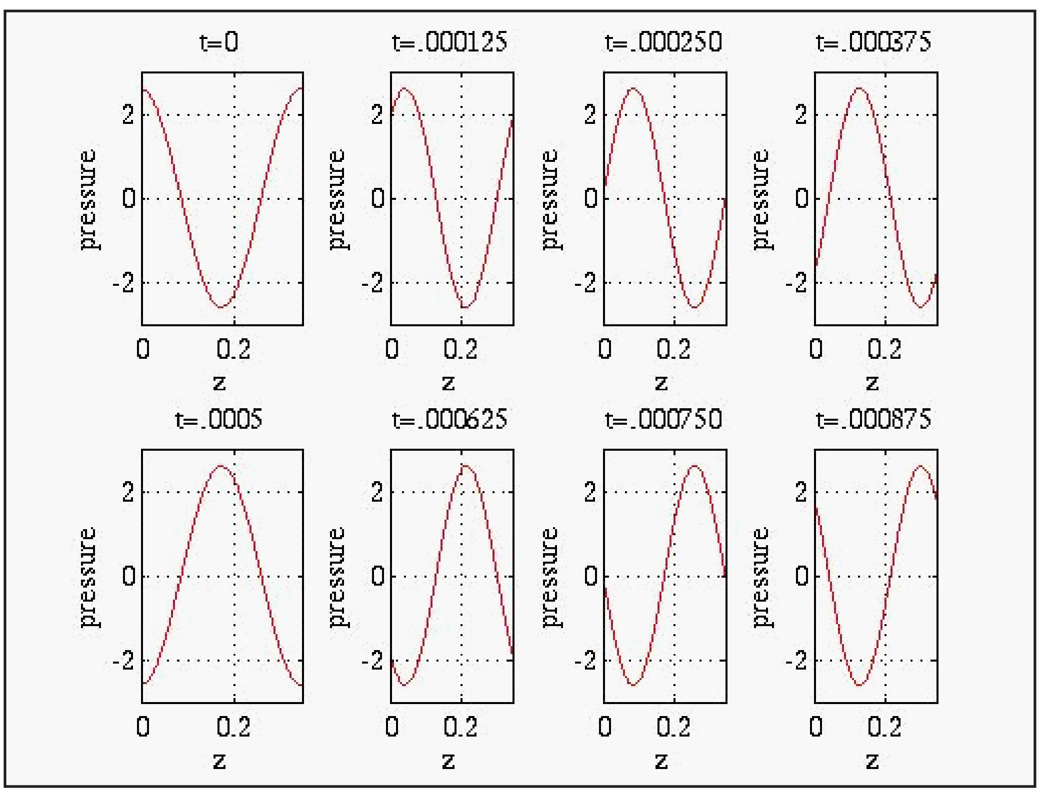


Figure 2. A depiction of successive shifts of the pressure waveform along the z-axis as time increases.

respectively, then

$$\mathbf{e} = \mathbf{e}_k + \mathbf{e}_p = \frac{1}{2}\rho_0 u^2 + \frac{1}{2}\frac{p^2}{\rho_0 c^2} = \frac{p^2}{\rho_0 c^2}$$

The last step in the above equation is justified because for a plane wave in air the particle velocity and acoustic pressure are related through $u = \frac{p}{\rho_0 c}$ thus making the kinetic and potential energy densities equal with each being one-half of the total energy density. The total acoustic energy density is also a function of position and time. Using the data from our numerical example the total acoustical energy density expression becomes

$$\mathbf{e} = 4.7374(10^{-5})\ \text{Joule}\bullet\text{m}^{-3}\bullet\cos^2\left(\frac{6{,}283}{\text{sec}}t - \frac{18.265}{\text{m}}z\right)$$

Fig. 4 is a plot of this energy density for a one-wavelength interval along the z-axis at the instant when t = 0.001 seconds. This corresponds to an elapsed time of one period of the piston motion.

Now we are in a position to calculate the total acoustic energy contained in the plane wave tube for a one-wavelength interval along the z-axis. If we draw a horizontal line across the peaks of the curve, we will now have a rectangle whose area numerically is (0.344)(4.7374)(10^{-5}). By visual inspection, however, the actual area beneath the curve indicated in blue is only 1 / 2 of this value so the average height of the curve is 1 / 2 of its peak value meaning that the average value of the energy density in this interval is 2.3687(10^{-5}) Joule·m^{-3}. (This is just an illustration of the fact that the average value of $\cos^2$ over one period is 1 / 2.) Here is the punch line. The total energy in the wave for this one-wavelength interval is the average energy density in the wave multiplied by the volume occupied by the wave. This is $S\lambda<\mathbf{e}>$ where $<\mathbf{e}>$ is the average value of the acoustic energy density. Calling this energy W, we have

$$W = S\lambda < \mathbf{e} > = \pi\left(\frac{.0254\text{m}}{2}\right)^2 (0.344\text{m})(2.3687)(10^{-5})\ \text{Joule}\bullet\text{m}^{-3}$$

$$= 4.1288(10^{-9})\ \text{Joule}$$

This is just the amount of energy supplied by the piston in the previous 0.001 second!

Next time we will investigate wave intensity and how to build a simple, practical plane wave tube. *ep*

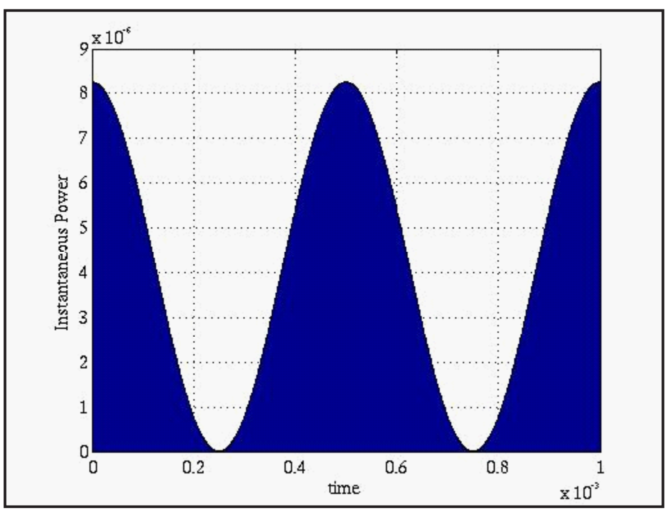


Figure 3. Instantaneous power delivered by the piston to the air in the plane wave tube. The blue area is the acoustic energy delivered to the sound wave in one period of the piston's motion.

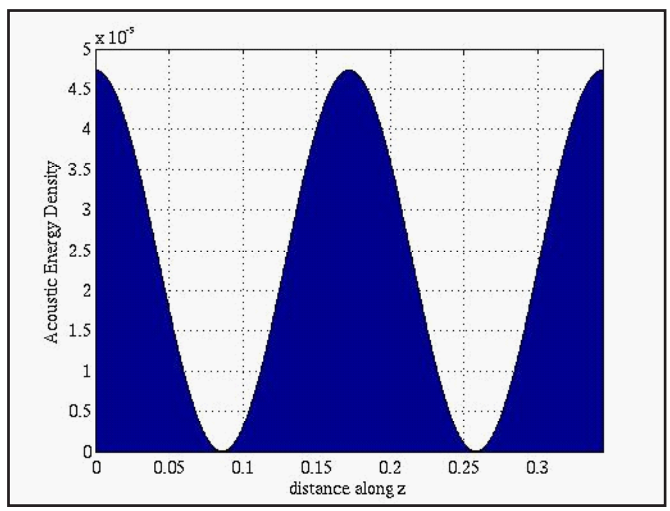


Figure 4. Acoustic energy density at t = 0.001 sec for a one-wavelength interval along the z-axis.