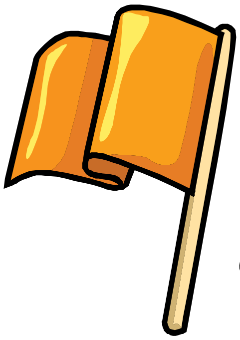
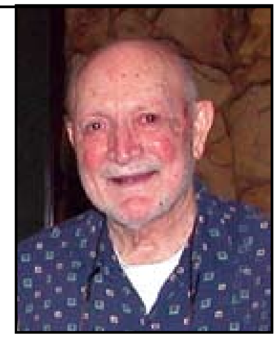


BY DR. EUGENE PATRONIS



What is Waving and Why?

The Anatomy of the Wave Equation - Part 2

In the first article in this series we learned the compositional properties of the gaseous mixture known as dry air as it exists under standard atmospheric conditions. We also explored the thermodynamic properties of air and the behavior of air while undergoing compression or expansion in either the isothermal or adiabatic process. We concluded the article with a sample calculation wherein the concept of acoustic pressure was introduced and used in the calculation of acoustic potential energy.

You will recall that the acoustic pressure is given by $p = P - P_0$. In this equation, p is the acoustic pressure at some point in space and some instant in time. Similarly, P is the disturbed total atmospheric pressure at the same point in space and the same instant in time while P_0 is the static or undisturbed atmospheric pressure at the location of interest.

The acoustic pressure is perhaps the premier acoustic variable. The root mean square value of the acoustic pressure at a particular location expressed in Pascals is what is used in determining the sound pressure level at that location through the relationship

$$SPL = 20dB\log\left(\frac{p_{rms}}{2(10^{-5})}\right)$$

There are, however, many other important acoustic variables whose values depend upon location in both space and in time. A listing of the ones to be employed in this article appears in Table 1.

Table 1 introduces three acoustic variables that we have not yet discussed. The acoustic condensation symbolized by s is simply the ratio of the change in air density brought about by an acoustic disturbance to the normal static or undisturbed air density as expressed by the equation

$$s = \frac{\rho - \rho_0}{\rho_0}$$

In this equation the Greek letter rho, ρ, represents the total density of air under the disturbed condition while ρ_0 represents the undisturbed or static air density. The acoustic variable that is termed the particle displacement and is symbolized by the Greek letter xi, ξ, will require a more lengthy explanation. The question that immediately arises is what constitutes an air particle? It cannot be a single molecule as air is always composed of a collection of a variety of molecules in the proportions tabulated in the first article of this series of articles. The particle size, whatever its value, must be sufficiently large so as to encompass millions of molecules in order to yield valid statistical averages and thus behave as an apparently continuous fluid while at the

Name of Variable	Symbol	Unit
Acoustic Pressure	p	Pascal
Air Density	ρ	$kg \cdot m^{-3}$
Condensation	s	dimensionless
Particle Displacement	ξ	m
Particle Velocity	u	$m \cdot sec^{-1}$

Table 1. A partial listing of acoustic variables.

same time it must be small enough that the acoustic variables are essentially constant throughout the volume occupied by the particle. This latter condition requires the dimensions of the particle to be very much smaller than any sound wavelength under consideration. Let's do a simple calculation in order to determine a reasonable size for what we will call an air particle. What volume would say two million molecules of air occupy under standard conditions? We learned in the first article that Avogadro's number of molecules would occupy about 0.0224 m^3 under standard conditions. If we consider a cube of edge dimension *l* then by simple proportion we can write

$$\frac{l^3}{2(10^6)} = \frac{0.0224}{6.02(10^{23})}$$

When this is solved for *l* the result is found to be $4.2(10^{-7})$ meter. This distance is orders of magnitude smaller than the wavelengths encountered in air even at ultrasonic frequencies so both of our requirements are satisfied. We might even round this number upward to a value easier to remember and say that an air particle is that amount of air under standard conditions that occupies a cube having an edge dimension of about 0.5 micron. When we consider such a small cubical volume of air that we now will call an air particle we realize that even in the absence of an acoustical disturbance, the air molecules are constantly undergoing random thermal motion. As a result of this thermal motion, some molecules move out of the volume but other molecules having the same properties also move into the volume. The volume has been chosen large enough so that the randomness of the thermal motion averages to zero so that in effect the air contained in the particle is at rest. An acoustical disturbance, as we shall see, imposes a preferred direction of motion and thus can bring about a displacement of the air particle as a whole. Particle displacement is a vector quantity and as such has both a magnitude and a direction that are measured relative to a coordinate frame of reference. In addition to particle displacement we will also be concerned with another acoustic variable that describes the instantaneous rate at which the particle displacement changes with time. This is a vector quantity also and is called the particle velocity. As the table indicates the symbol employed for the particle velocity is the letter u.

In the absence of any acoustical disturbance the acoustic variables of Table 1 are all zero with the exception of ρ for which the value becomes the static atmosphere value ρ_0. When an acoustical disturbance is present all of the acoustic variables listed in the table will have values that depend upon both location in space and time. For simplicity let's center our attention on just the acoustic pressure as an example. Mathematically we say that the acoustic pressure is a function of the positional coordinates and time. If we were employing general Cartesian coordinates this mathematical statement would be written in the manner,

$$p = p(x,y,z,t).$$

The entry immediately above is read as, "The acoustic pressure is a function of x, y, z, and t." It does not tell you what particular mathematical function but only that there is such a function. In certain situations not all spatial coordinates may be involved. If the acoustic pressure depends only on the z coordinate and time then $p = p(z,t)$ would be appropriate. From either theory or experiment we may find what the particular mathematical functional dependence is. For example, the answer might be

$$p = p_m \cos(\omega t - kz).$$

In this answer p_m, ω, and k are constants and we are informed that the acoustic pressure varies as the cosine of the difference of two angles one of which is directly proportional to time and the other of which is directly proportional to the value of the z coordinate. As we shall see shortly this function describes a plane wave propagating in the direction of increasing values of the z co-

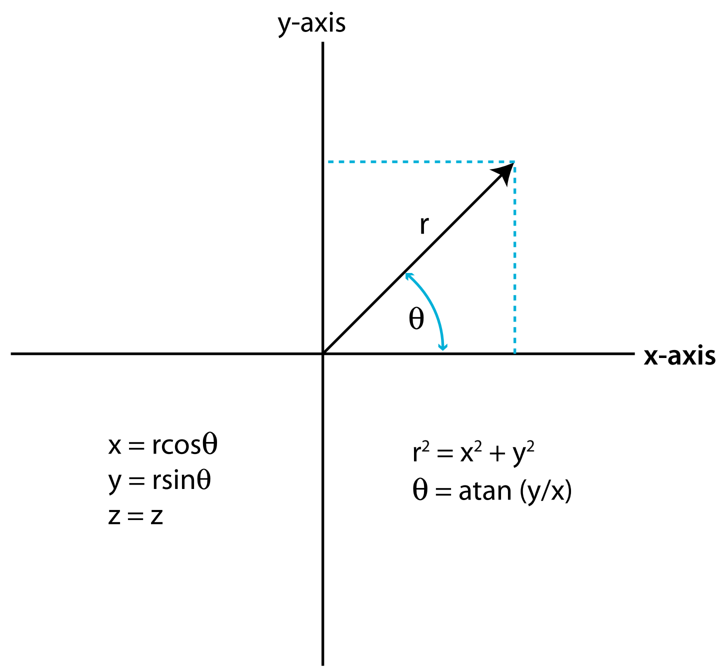


Figure 1. Cartesian to cylindrical conversion. The z axis points toward the reader.

ordinate.

Now that we have covered the preliminaries, we turn our attention to a physical system of some importance consisting of a long, rigid-wall air filled pipe. The inner diameter of the pipe is d, its inner radius is a. The interior wall is smooth and the pipe is straight. The wall thickness of the pipe is immaterial as long as it is reasonably rigid. We will employ cylindrical coordinates for locating positions in the pipe. These are the coordinates best suited for such a structure. In cylindrical coordinates space points are located by the variables r, θ and z. The z coordinate is familiar from the usual Cartesian set. The relationship between r, θ and the familiar x, y can be extracted by viewing Fig. 1.

We have selected an air filled pipe as the starting point for our discussion of acoustic waves because of the ease with which the simplest of wave motions, namely plane waves, can be established in such a structure. Our first step will be to concentrate on a small mass of air in the pipe under static conditions and on the same mass of air after it has been acoustically disturbed. The physical situation is depicted in Fig.2.

The pipe has an inner cross-sectional area $S = \pi a^2$. The undisturbed air is that contained in the cylindrical volume between the planes defined by z and z + Δz. The mass, m, of this air is the static air density multiplied by the volume of the blue cylinder.

$$m = \rho_0 S\Delta z$$

Imagine now that a closely fitting piston is inserted into the pipe on the left and quickly displaces the air particles that were originally on the plane at z to the new red planar position z + ξ such that all of the air particles that were originally at z are now located at z + ξ. In other words, the air particles originally located at the spatial coordinate z have undergone an amount of displacement equal to ξ. Note also that the particle displacement does not depend on the spatial coordinates r or θ. All air particles having a particular value of the z coordinate are displaced the same amount such that the particle displacement depends only on z and on time, t. Now if air were incompressible, all of the particles originally on the plane at z + Δz would be displaced to a new plane at z + Δz + ξ. Air is compressible however, so we must allow for the particle displacement to undergo a change over the space interval of Δz so that the right extremity of our disturbed mass of air is located at z + Δz + ξ + Δξ. Our original mass of air is now contained in the cylinder defined by the two red planes. The air has been compressed as a result of the piston motion. As a consequence, the volume of the disturbed cylinder is slightly less than that of the undisturbed one and this simply requires that Δξ be a negative number. The mass of air was conserved in the process so the density of air in the disturbed cylinder has increased. The volume of the undisturbed cylinder is SΔz while that of the disturbed cylinder is S(Δz + Δξ) so we may write

$$r_0 S\Delta z = r\, S(\Delta z + \Delta x) = r\, S\Delta z\left(1 + \frac{\Delta x}{\Delta z}\right)$$

This equation readily simplifies to

$$r_0 = r\left(1 + \frac{\Delta x}{\Delta z}\right)$$

In words this last equation says that the undisturbed density of air is equal to the disturbed density multiplied by one plus the average slope of the particle displacement function over the interval Δz. This slope is negative however as the particle displacement decreases as z increases and thus the number in the parenthesis is less than one. The average slope is not good enough. We need

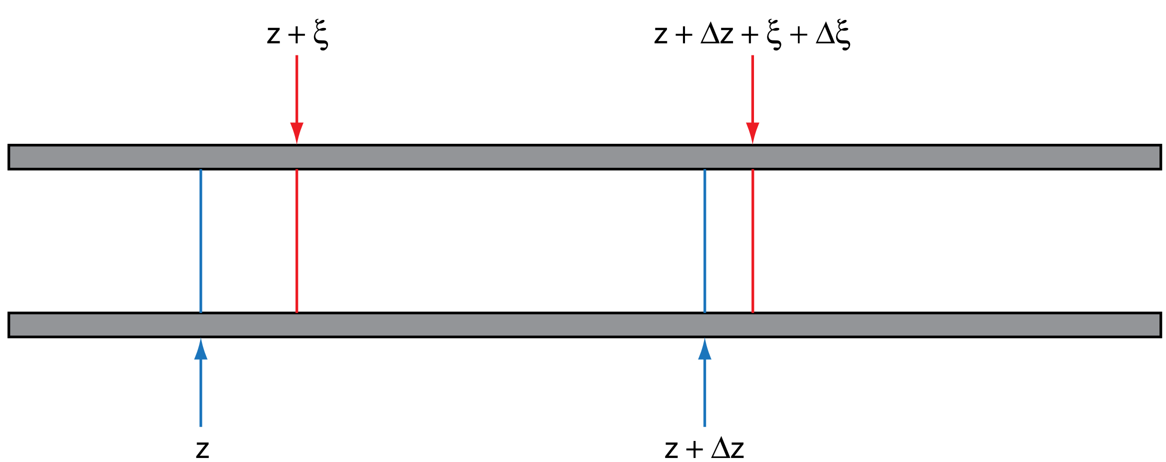


Figure 2. Undisturbed (blue) and disturbed (red) air mass in a long rigid pipe.

to make our calculation independent of our choice of the size of Δz. At this point I recognize that many readers have not had an opportunity to study calculus much less partial differential equations. Both of these are required in order to do a rigorous derivation of the wave equation. In much of the following then, I will substitute word descriptions for what is going on rather than adhering to pure mathematical formalism. In the density equation above we take successively smaller and smaller sizes for Δz or in other words let Δz approach zero all the while studying the ratio Δξ/Δz and look to see what limiting value is approached by the quotient of Δξ/Δz. This limit is called the partial derivative of the particle displacement with respect to the z coordinate and the density relation is then written as

$$\rho_0 = \rho\left(1+\frac{\partial\xi}{\partial z}\right)$$

The reason for doing this is to find the value of the disturbed air density in the immediate vicinity of the point z and at time t. Among other things, we want to learn how the air density behaves under disturbed conditions as a function of position and time. This last equation tells us how to calculate the density behavior once we know how the particle displacement behaves. Recall that the condensation is given by

$$s = \frac{\rho - \rho_0}{\rho_0}$$

This can be solved for the disturbed density to yield $\rho = \rho_0(1 + s)$. This is now substituted in the density relation to yield

$$\rho_0 = \rho_0(1+s)\left(1+\frac{\partial\xi}{\partial z}\right)$$

$$1 = (1+s)\left(1+\frac{\partial\xi}{\partial z}\right)$$

$$1 = 1+\frac{\partial\xi}{\partial z}+s+s\frac{\partial\xi}{\partial z}$$

In order for the remainder of our development to be as simple as possible we must restrict the size of the acoustical disturbance to that for which the air behaves as a linear medium. Even with this restriction the equations that we develop will accommodate sound pressure levels up to 120 dB with little error. With this restriction, we can observe that in the last equation written above both s and the partial derivative of the displacement with respect to z are small quantities individually and that the product of the two of them is very small indeed. Hence neglecting the product term introduces negligible error. This final equation can then be rewritten as

$$s = -\frac{\partial\xi}{\partial z}$$

In the first article in this series we learned that for small disturbances the acoustic pressure is given by

$$p = c^2(\rho - \rho_0)$$

and in terms of the condensation this may be written as

$$p = \rho_0 c^2 s$$

Alternatively, we may express the acoustic pressure as

$$p = -\rho_0 c^2 \frac{\partial\xi}{\partial z}$$

One other observation is appropriate at this point. The particle displacement, ξ, is in general a function of both z and t. This means that ξ = ξ(z,t). In fact, one of our objectives is to find the exact nature of this function for a given type of acoustical excitation. Once we determine the nature of this function we can determine the value of the particle displacement for any value of the spatial coordinate z and time coordinate t. Additionally we will also be able to determine the particle velocity, u, as the particle velocity at any particular value of the z coordinate and time t is the rate at which the particle displacement is changing with time at the fixed value of z. The particle velocity is given by the partial derivative of the particle displacement with respect to time and is given by

$$u = \frac{\partial\xi}{\partial t}$$

Similarly, the local particle acceleration or rate of change of velocity is calculated from

$$\frac{\partial u}{\partial t} = \frac{\partial^2\xi}{\partial t^2}$$

Thus far we have required only a few definitions, the law of conservation of mass, and knowledge of the behavior of air while undergoing small adiabatic compression or expansion. Now with the help of Sir Isaac Newton's second law of motion we will be able to finally arrive at the plane wave equation. First we must list the forces that could possibly affect the air particle motion in the tube. The principal force is that exerted by the piston as it first begins to compress the air at the left face of our undisturbed cylinder of air as depicted in Fig. 3. The pressure exerted by the piston must exceed static atmospheric pressure in order to produce compression so we write this as $(P_0 + p)S$ where p is the acoustic pressure at z. Similarly, the pressure at the right face is the static pressure plus the acoustic pressure at the right face which we must allow to be different from that at the left face. We write the force at the right face then as $(P_0 + p + \Delta p)S$. In principle, the force of gravity would tend to make the static pressure at the bottom of the tube minutely greater than that at the top. This effect is insignificant for tubes of ordinary diameters. Finally, we should mention the possibility of viscous effects. Viscous frictional forces occur principally at the tube walls and manifest themselves as a small attenuation in very long tubes. We will neglect such effects for reasons of simplicity.

From Fig. 3, the net force in the positive z direction is - ΔpS. According to Newton's second law the net force acting on the mass of air in the element must be equated to the mass multiplied by the acceleration or

$$-\frac{\Delta p}{\Delta z}\Delta z S = \rho_0 \Delta z S \frac{\partial^2 \xi}{\partial t^2}$$

Upon canceling common factors and taking the limit as we have done in a previous case this equation becomes

$$-\frac{\partial p}{\partial z} = \rho_0 \frac{\partial^2 \xi}{\partial t^2}$$

In words this result says that the negative of the space rate of change of the acoustic pressure at a given point and time is the undisturbed density of air multiplied by the particle acceleration at the same space point and time t.

Two more steps and we will be at the punch line. One of our previous results while studying the particle displacement was

$$p = -\rho_0 c^2 \frac{\partial \xi}{\partial z}$$

From this relation we need to calculate the space rate of change or the slope of the acoustic pressure. This is done by calculating the partial derivative with respect to the z coordinate on both sides of the equation. The result is

$$\frac{\partial p}{\partial z} = -\rho_0 c^2 \frac{\partial^2 \xi}{\partial z^2}$$

This is now substituted into the equation derived employing Newton's second law to produce

$$\frac{\partial^2 \xi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}$$

This second order partial differential equation is the governing equation for plane waves that depend on only one space coordinate and time as the independent variables. The dependent variable in this instance is the air particle displacement. Instead of concentrating on the particle displacement as the dependent variable, we could have just as well done a parallel development while centering our attention on the acoustic pressure to obtain

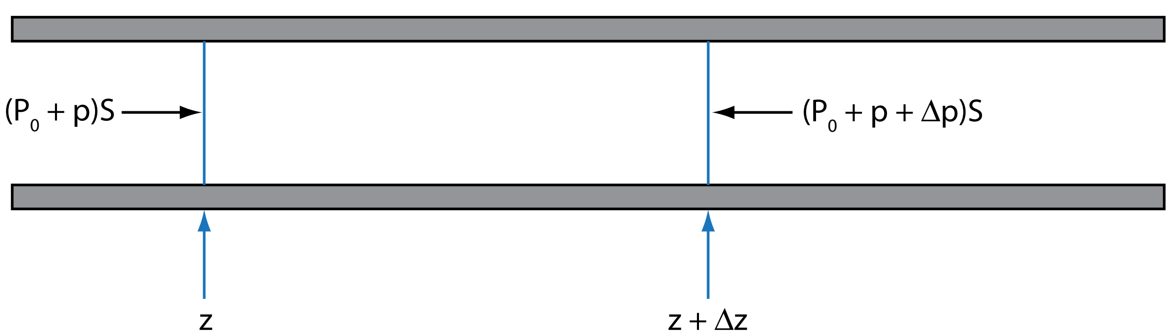


Figure 3. Forces acting on undisturbed element at the onset of compression by the piston. S is the cross-sectional area of the pipe.

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

In other words, the acoustic pressure and the particle displacement are governed by the same partial differential equation. In order for some mathematical function to be a solution to a physical circumstance involving the plane wave equation it must accomplish three things. Firstly, when substituted into the wave equation it must yield an identity. Secondly, it must satisfy the conditions that exist at t = 0. Finally, it must satisfy the conditions that exist at the coordinate boundaries for all values of t ≥ 0. There are many functions that satisfy the first condition. In fact, there are an infinite number of such functions. All of these functions, however, have one feature in common and that is whenever the space and time independent variables appear in one of the functions, this appearance must be of the form (ct ± z). The two other requirements play the role of sorting through this infinite set to find the one and only solution that fits the problem at hand. We are assured that there is only one genuine solution to the wave equation that satisfies the three stated requirements because of the existence of a uniqueness theorem governing solutions to the wave equation. In order to make this really meaningful, we must seek a solution to this equation for a realizable physical circumstance. First, let's illustrate the significance of (ct ± z). Suppose we have a very long plane wave tube with the origin of coordinates at the mid-point of the tube. Further suppose at t = 0 that some disturbance produces an acoustic pressure matching only the blue curve in Fig. 4.

Refer now to Fig. 4 and imagine that only the blue curve is present. This would represent the first frame of a movie describing the acoustic pressure versus time and position in space. The second frame would show the initial disturbance beginning to split into two equal parts with one part displaced slightly to the left and the other displaced an equal amount to the right. Many frames later, the blue curve would no longer be present and the two red curves would represent a snapshot at the instant when ct = 4 meters. In other words the initial static disturbance has evolved into two traveling disturbances moving in opposite directions along the z-axis. The functions involved would be $p_+ = p(ct - z)$ and $p_- = p(ct + z)$. Let's concentrate on just the p_+ term. If we are to always observe the same peak pressure for this term, what must we do as an observer? Remember that we have no control over time. It increases uniformly whether we want it to or not. As ct increases uniformly then we must increase our location on the z-axis at the same rate such that ct – z maintains a value of, in this case, zero. The rate at which ct increases is the speed of sound, c. Therefore an observer must race in the direction of increasing z with a speed equal to c in order to keep up with the pressure pulse on the right. Similarly, an observer must race in the direction of decreasing z with a speed c in order to keep up with the pressure pulse on the left.

In part 3 of this series we will solve the wave equation for a plane wave tube excited by a periodic piston motion at one end and study the acoustical energy transport in the tube. *ep*

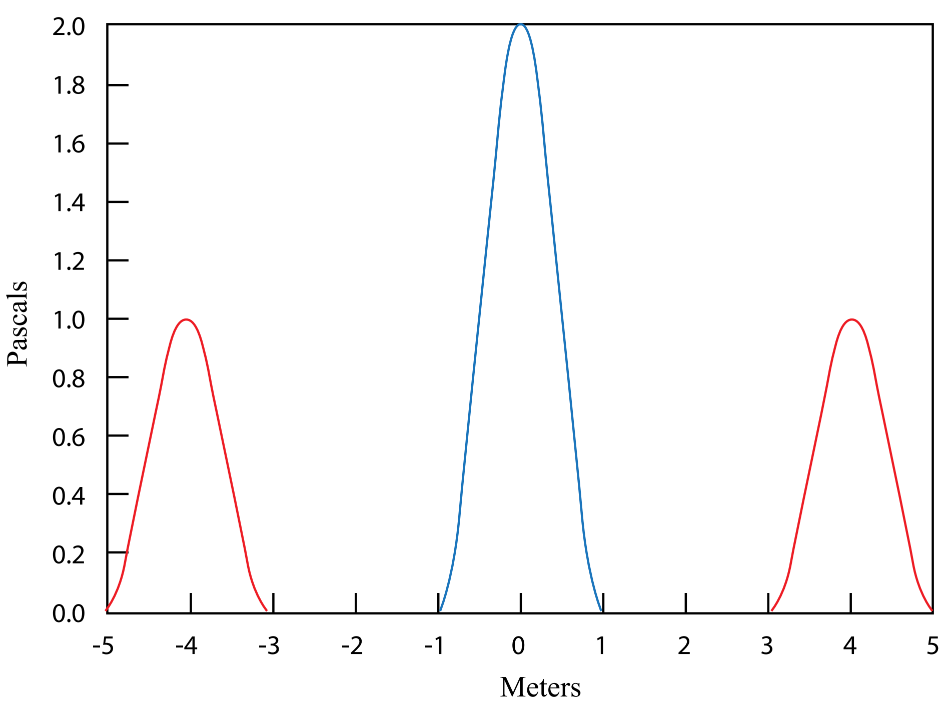


Figure 4. Plane wave tube with an initial disturbance (blue) at its center.